Point-countable Covers and Sequence-covering Mappings

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Abstract

The method of mappings and covers is a basic tool to study general topology. The theory of generalized metric properties and covering properties stems from a generalization of metrizability and compactness, in which a great many problems involve a research for certain point-countable covers. A deliberation on questions related to point-countable covers leads to a development of the theories of k-networks and mappings of metric spaces in generalized metric spaces. This book summarizes the essential tasks and the important contribution to the theory of generalized metric spaces in the past ten years, and in four chapters expounds some results about point-countable covers and sequence-covering mappings from 1993 to 2000.

In the second chapter, we centre on point-countable covers and the theory of s-mappings of metric spaces. In the third chapter, we present a sequence of point-finite covers and the theory of π-mappings and compact mappings of metric spaces. In the fourth chapter, we discuss hereditarily closure-preserving covers, dominated families and the theory of closed mappings of metric spaces. In the fifth chapter, we study star-countable covers and the theory of quotient mappings of locally separable metric spaces.

Contents

A Foreword

The notions of metrizability and compactness are at the center of general topology. This is so from historical point of view: general topology developed around these notions. This is also so if we look at the structure of general topology today. Compactness is defined in terms of open coverings. It turned out, after the classic work of P. S. Alexandroff, P. S. Urysohn, F. B. Jones, A. H. Stone, R. H. Bing, J. Nagata, and Ju. M. Smirnov, that metrizability can be also characterized in terms of sequences of open coverings. Nowadays it is widely recognized that the method of systems of coverings is one of the maim tools for classifying spaces. Many most important classes of spaces were introduced using natural covering structures which find their origins in metrizable spaces. In fifties and sixties it was discovered that systems of coverings can be used very effectively to construct some natural mappings of metrizable spaces onto spaces admitting such systems of coverings (K. Nagami, V. I. Ponomarev, A. Okuyama, A. Arhangel'skii). This method led to mutual classification of spaces and mappings based on the interaction of systems of coverings and mappings(see [Arhangel'skii, 1966]).

This monograph contains a systematic study of classes of generalized metric spaces on the basis of coverings and networks, in combination with the general method of mappings. It presents the modern state of the mutual classification of spaces and mappings, one of central domains of general topology. The author, professor Shou Lin, and his collaborators and colleagues, first of all, from China(Chuan Liu, Pengfei Yan, Mumin Dai) and Japan(K. Nagami, Y. Tanaka, T. Hoshina), made major contributions to this theory with important, and often quite unexpected, results.

An important feature of the book is that it may serve as a systematic source of information on classification of classes of mappings and on methods used to construct mappings with special combinations of properties. This direction of research is not sufficiently well covered in the existing textbooks and this monograph will be especially valuable in this connection. However, the book will be very useful to all general topologists, even to those who are working on very different problems, and to graduate students in general topology with various interests. Indeed, the monograph provides an interesting and upto date introduction to metrizability and to classes of generalized metric spaces. It introduced basic classes of mappings, in particular, quotient mappings, compact covering mappings and to various generalizations of those. These notions are of very fundamental nature.

Another notion of fundamental nature, studied and applied systematically in the monograph, is the notion of network introduced in 1959. A family $\mathcal{S}$ of subsets of a topological space X is called a network of X if every open subset is the union of a subfamily of $\mathcal{S}$. The networks were used in [Arhangel'skii, 1959] to distinguish the spaces with a countable network. It was observed in [Arhangel'skii, 1959] that every continuous image of a separable metric space has a countable network, and it was also established in [Arhangel'skii, 1959] that every compact Hausdorff space with a countable network has a countable base(and is therefore metrizable). From these statements it follows that if a compact Hausdorff space Y is a continuous image of a separable metrizable space, then Y is metrizable. This theorem from [Arhangel'skii, 1959] is one of the first and typical results in the general theory of spaces and mappings the modern state of which is presented in this monograph. It turned out that networks are much more subtle, more flexible structures than bases, which are "too nice formations". It is through various, often ingenious, combinations of restrictions on networks that most important classes of generalized metric spaces are introduced and studied. One of such classes was defined by E. Michael, using the concept of k-network[Michael, 1966], which is really central in the monograph.

Though reading the monograph does no require much special background, the exposition in it goes far beyond the elementary level. It contains a rich collection of deep and beautiful results of highest professional level. Theorem 2.1.9, Corollary 2.1.11, and Theorems 2.2.5, 2.2.10, 2.3.11, 2.3.13 and Theorem 2.5.10 can serve as examples of such excellent results. In particular, these theorems provide sufficient conditions for a mapping to be compact covering and sufficient conditions for a space to be a quotient space of a metric space with separable fibres.

The book not only brings the reader to the very first line of investigations in the theory of generalized metric spaces, it contains many intriguing and important unsolved problems, some of them old and some new. Here is one of the open problems, a very natural one. Suppose Y is a quotient space of a metrizable space, with compact fibres. Suppose further that Y is regular. Is then every point in Y an intersection of a countable family of open sets? This book provides all background and information needed to start to work on this and similar problems. So the monograph will be equally interesting and useful for an expert and for a graduate student.

I would like to mention another joyful aspect of this monograph. Its appearance marks the success of a long period of development of general topology in China, it brings to the light important contributions to the mainstream of general topology made by a very creative group of Chinese mathematicians.

Congratulations to the author and to the readers with the book!

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Chapter 1 Introductions

1.1 Recent advances in generalized metric spaces

Metrization

Compact-coverings mappings and related matters

Product spaces

Monotone normal spaces and M_i-spaces

Bases, k-networks and networks

1.2 Nations and terminology

All spaces are T_2, mappings are continuous and onto. $<x_n>=\{x_n : n\in N\}$, τ(X) or τ denotes a topology of a space X.

Definition 1.2.3 Let $f:X\to Y$ be a mappping.

(1)(Michael[1966]) f is compact-covering if each compact subset of Y is the image of some compact subset of X.

(2)(Boone-Siwiec[1976]) f is suquentially quotient if whenever $\{y_n\}$ is a convergent sequence in Y, there are a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and a convergent sequence $\{x_i\}$ in X with each $x_i \in f^{-1}(y_{n_i})$.

(3)(Siwiec[1971]) f is sequence-covering if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$.

(4)(Gruenhage-Michael-Tanaka[1984], Ikeda-Liu-Tanaka[2001]) f is pseudo-sequence-covering if each convergent sequence(containing its limit point) in Y is the image of some compact subset of X.

(5)(Lin-Liu-Dai[1997]) f is subsequence-covering if whenever $\{y_n\}$ is convergent sequence in Y, there is a compact subset K in Y such that f(K) is a subsequence of $\{y_n\}$.

(6)(Lin[1996c]) f is 1-sequence-covering if there is $x\in f^{-1}(y)$ for each $y\in Y$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(7)(Lin[1996c]) f is 2-sequence-covering if $y\in Y$ and $x\in f^{-1}(y)$, whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

Definition 1.2.7 Let $\mathcal{P}$ be a cover of a space X.

(1)(Arhangel'skii[1959]) $\mathcal{P}$ is a network if whenever $x\in U$ with U open in X, then $x\in P\subset U$ for some $P\in\mathcal{P}$.

(2)(O'Meara[1971]) $\mathcal{P}$ is a k-network if whenever $K\subset V$ with K compact and V open in X, then

$K \subset \cup \mathcal{P}' \subset V$ for some $\mathcal{P}' \in \mathcal{P}^{<\omega}$.

(3)(Guthrie[1971]) $\mathcal{P}$ is a cs-network if whenever a sequence $\{x_n\}$ converges $x \in V$ with V open in X, then $\{x_n\}$ is eventually P and $P \subset V$ for some $P \in \mathcal{P}$.

(4)(Gao[1987]) $\mathcal{P}$ is a cs*-network if whenever a sequence $\{x_n\}$ converges $x \in V$ with V open in X, then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is eventually P and $P \subset V$ for some $P \in \mathcal{P}$.

Definition 1.2.8 Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a family of subsets of a space X such that for each $x \in X$, $\mathcal{P}_x$ is a network of x in X, and if U, $V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1)(Lin[1996c]) $\mathcal{P}$ is a sn-network if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X.

(2)(Lin[1996c]) $\mathcal{P}$ is a so-network if each element of $\mathcal{P}_x$ is sequential open in X.

(3)(Arhangel'skii[1966]) $\mathcal{P}$ is a weak base if $G \subset X$ such that for each $x \in G$, there is $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in X.

$\mathcal{P}_x$ is a sn-network of x, a so-network of x and a weak base(or, weakly neighborhood base, weakly neighborhood network) of x in X, respectively.

(4)(Arhangel'skii[1966]) X is a gf-coutable space if each point of X has a countable weak base.

cosmic space(Michael[1966])=A space with a countable network.

$\aleph_0$-space(Michael[1966])=A space with a countable k-network.

g-metrizable space(Siwiec[1974])=A regular space with a σ-locally finite weak base.

1.3 Prerequisites: generalized metric spaces and images of metric spaces

1.4 Prerequisites: quotient mappings and weakly first countability

1.5 Examples

Chapter 2 On point-countable covers

In this chapter we centre on point-countable covers and the theory of s-mappings of metric spaces, give some answers to questions posed by A. Arhangel'skii, T. Hoshina, Chuan Liu, E. Michael, K. Nagami, Chuan Liu, Y. Tanaka etc.(see. Question 2.1.1, Question 2.2.1, Question 2.3.2, Question 2.4.1

and Question 2.5.1), and establish several models of the s-images of sequence-covering mappings and compact-covering mappings of metric spaces.

2.1 wcs*-networks and bases

In this section we give affirmative answer about Liu and Tanaka's questions.

Question 2.1.1 (1)(Liu-Tanaka[1998b], Tanaka[1994]) Let X be a k-space with a point-countable k-network. Does X has a point-countable base if X contains no closed copy of S_2 and S_ω ?

(2)(Tanaka[1987a]) Let X be a quotient s-image of a metric space. Does X has a point-countable base if X contains no closed copy of S_2 and S_ω ?

(3)(Tanaka[1994]) Let X be a sequential space with a point-countable cs-network. Does X has a point-countable weak base if X contains no closed copy of S_ω ?

(4)(Liu-Tanaka[1996a]) Let X be a k-space with a σ-point-finite k-network. Is X a gf-countable space if X contains no closed copy of S_ω ?

Definition 2.1.2(Lin-Tanaka[1994]) Let $\mathcal{P}$ be a cover of a space X. $\mathcal{P}$ is a wcs*-network if whenever $\{x_n\}$ is a sequence converging to x in X and $x\in U\in \tau$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $<x_{n_i}> \subset P \subset U$ for some $P\in \mathcal{P}$.

Open question 2.1.8(Lin-Yan[1998]) Let X be a regular and Fréchet space with a point-countable k-network. Does X has a point-countable cs*-network if X contains no closed copy of S_{ω_1}?

Let (X, τ) be a space. σX denotes a set X with a topology σ_τ as follows. $U\in\sigma_\tau$ iff U is sequential open in τ .

Theorem 2.1.9 Let X be a space with a point-countable wcs*-network. Then

(1) σX is a Fréchet space if σX contains no closed copy of S_2 .

(2) σX is a gf-countable space if σX contains no closed copy of S_ω .

The following corollaries answer all questions in 2.1.1.

Corollary 2.1.11 Let X is a sequential space with a point-countable wcs*-network. Then

(1) X is a Fréchet space iff X contains no closed copy of S_2 .

(2) X is a gf-countable space iff X contains no closed copy of S_ω .

(3) X is a first countable space iff X contains no closed copy of S_2 and S_ω .

If X is still a regular space, then

(4) X has a point-countable base iff X contains no closed copy of S_2 and S_ω .

Corollary 2.1.12 (1) X has a point-countable sn-network iff X has a point-countable cs-network and σX contains no closed copy of S_ω .

(2) X has point-countable weak base iff X is a sequential space with a point-countable cs-network, and X contains no closed copy of S_ω .

(3) X has point-countable base iff X is a sequential space with a point-countable cs*-network, and X contains no closed copy of S_2 and S_ω .

Theorem 2.1.15 If X is a regular and Fréchet space with a point-countable wcs*-network, then X is a hereditarily metalindelöf space.

Open question 2.1.17 (1)(Liu-Tanaka[1996b]) Let X is a sequential space with a point-countable cs*-network. Does X has a point-countable cs-network if X contains no closed copy of S_2 ?

(2) Find a nice mapping such that a space with a point-countable wcs*-network can be characterized by the image of metric spaces under the mapping.

2.2 k-networks and closed mappings

In this section we present some mapping theorems about spaces with a point-countable k-network.

Question 2.2.1(Lin-Tanaka[1994]) Are the spaces with a point-countable k-network preserved by closed mappings?

Example 2.2.2(Sakai[1997b]) There is a spaces with a point-countable k-network which can not be preserved by closed mappings.

Theorem 2.2.5(Lin-Tanaka[1994]) Let $f:X\to Y$ be a closed mapping, and X has a point-countable k-network. If one of the following conditions holds, then Y has a point-countable k-network.

(1) X is a k-space.

(2) X is a regular space with a point-G_δ property.

(3) X is a normal and isocompact space.

(4) X is a regular space and each $\partial f^{-1}(y)$ is a Lindelöf subset of X.

Corollary 2.2.6 Each regular space with a point-countable base is preserved by open and closed mappings.

Conjecture 2.2.7 Let $f:X \to Y$ be a compact-covering and closed mapping. If X has a point-countable k-network, then Y has a point-countable k-network iff each compact subset of Y is metrizable.

Let $f:X \to Y$ be a closed mapping. If X is a normal and isocompact space, or a regular space such that each $\partial f^{-1}(y)$ is a Lindelöf subset of X, then f is compact-covering.

Theorem 2.2.10(Lin-Liu[1996], Shibakov[1995b]) Let $f:X \to Y$ be a closed mapping, and X be a regular space with a point-countable k-network. If one of the following conditions holds, then f is compact-covering.

(1) X is a k-space.

(2) X has a point-G_δ property.

Open question 2.2.12(Sakai[1997b]) Is each space a closed image of a space with a point-countable k-network?

2.3 Sequential networks and quotient mappings

In this section we introduce the concept of sequential networks, establish some relations between sequential networks and cs*-networks, and obtain new characterizations of quotient images of metric spaces.

About quotient images of metric spaces, Arhangel'skii posed the following questions.

Question 2.3.2 (1)(Svetlichny[1993]) Is a sequential space with weight κ a quotient image of a metric space with weight κ ?

(2)(Arhangel'skii[1966]) Find an internal characterization of quotient s-images of metric spaces.

Definition 2.3.3(Lin[1999a]) Let $\mathcal{P}$ be a family of subsets of a space X such that there is $\mathcal{P}_x \subset \mathcal{P}^\omega$ satisfying that $\langle P_n \rangle$ is a decreasing network of x in X if $(P_n) \in \mathcal{P}_x$ for each $x \in X$.

(1) $\mathcal{P}$ is a sequential network if $P \subset X$ such that there is $m \in N$ with $P_m \subset P$ for each $(P_n) \in \mathcal{P}_x$ and

each $x \in P$, then P is sequential open in X.

(2) $\mathcal{P}$ is a sequential quasi-base if $P \subset X$ such that there is $m \in N$ with $P_m \subset P$ for each $(P_n) \in \mathcal{P}_x$ and each $x \in P$, then P is open in X.

Theorem 2.3.11(Lin[1999a]) For each cardinal κ and each space X, X has a sequential network with cardinal κ iff X is a sequentially quotient image of a metric space with weight κ.

Corollary 2.3.12(Svetlichny[1993]) For each cardinal κ and each space X, X has a sequential quasi-base with cardinal κ iff X is a quotient image of a metric space with weight κ.

Theorem 2.3.13(Lin[1999a]) The following are equivalent for a space X:

(1) X has a point-countable sequential network.

(2) X has a point-countable cs*-network.

(3) X is a sequentially quotient s-image of a metric space.

Open question 2.3.18 Let X be a space with a point-countable base. Is there a quotient s-mapping $f:M \to X$ with M metric such that each $|\partial f^{-1}(x)| \leq 1$?

2.4 cs-networks, sn-networks and sequence-covering mappings

In this section we study some characterizations of s-images of sequence-covering(or, 1-sequence-covering, 2-sequence-covering) mappings of metric spaces.

Question 2.4.1(Hoshina[1970]) Find a nice mapping such that a space with a point-countable weak base can be characterized by the image of metric spaces under the mappings.

Theorem 2.4.2(Woods[1979]) Each regular space is a prefect image of a completely regular space with a point-countable so-network.

Theorem 2.4.3(Lin[1996b]) A space X is a sequence-covering s-image of a metric space iff X has a point-countable cs-network.

Theorem 2.4.6(Lin[1996c]) A space X is a 1-sequence-covering s-image of a metric space iff X has a point-countable sn-network.

Corollary 2.4.7 A space X is a 1-sequence-covering quotient and s-image of a metric space iff X has a point-countable weak base.

Theorem 2.4.9(Lin[1996c]) A space X is a 2-sequence-covering s-image of a metric space iff X has a point-countable so-network.

Theorem 2.4.15(Lin[2000a]) Let $f:X\rightarrow Y$ be a mapping. If X is a first countable space, then

(1) f is open iff f is a 2-sequence-covering and quotient mapping.

(2) f is almost open iff f is a 1-sequence-covering and pseudo-open mapping.

Open question 2.4.16(Tanaka[1993]) Each gf-countable space can be preserved by open and closed mappings?

2.5 cfp-networks and compact-covering mappings

In this section we introduce the concept of cfp-networks, and obtain an internal characterization of compact-covering s-images of metric spaces.

Question 2.5.1(Michael-Nagami[1973]) Is a quotient s-image of a metric space a compact-covering quotient s-image of a metric space?

Example 2.5.2(Chen[1999]) There is a quotient and compact image X of a locally separable metric space such that X is not any compact-covering quotient and s-image of a metric space.

Definition 2.5.3(Yan[1997]) Let K be a subset of a space X. $\mathcal{F}$ is a cfp-cover of K if $\mathcal{F}$ is a finite cover of K in X and is refined one-to-one by a finite cover of compact subsets of K.

Suppose that $\mathcal{P}$ be a cover of a space X.

(1)(Yan[1997]) $\mathcal{P}$ is a compact-finite-partition of X if for each compact subset K of X there is a $\mathcal{P}' \in \mathcal{P}^{<\omega}$ such that $\mathcal{P}'$ is a cfp-cover of K.

(2)(Liu-Dai[1996]) $\mathcal{P}$ is a strong k-network if for each compact subset K of X there is a countable subset $\mathcal{P}(K)$ of $\mathcal{P}$ such that for each compact subset L of K and open subset $V\supset L$, there is $\mathcal{P}' \in \mathcal{P}(K)^{<\omega}$ such that $\mathcal{P}'$ is a cfp-cover of L and $\cup\mathcal{P}' \subset V$.

(3)(Yan-Lin[1999b]) $\mathcal{P}$ is a cpf-network if for each compact subset K and open subset $V\supset K$, there is $\mathcal{P}' \in \mathcal{P}^{<\omega}$ such that $\mathcal{P}'$ is a cfp-cover of K and $\cup\mathcal{P}' \subset V$.

Theorem 2.5.9 The following are equivalent for a space X:

(1) X is a compact-covering s-image of a metric space.

(2)(Yan-Lin[1999b]) X has a point-countable cfp-network.

(3)(Liu-Dai[1996]) X has a point-countable strong k-network.

Open question 2.5.12 (1) Does a sequential space with a compact-countable cs*-network has a point-countable cfp-network?

(2) Is a regular space with a point-countable cs*-network a space with a point-countable cfp-network?

Chapter 3 Sequences of point-finite covers

In this chapter we introduce a sequence of point-finite covers of spaces, establish the theory of π-mappings and compact mappings of metric spaces, and give some characterizations of sequence-covering compact images of metric spaces.

3.1 Point-star networks and π-mappings

In this section we introduce the concept of point-star networks of a space, and discuss some relations between point-star networks and sequence-covering π-images of metric spaces.

Definition 3.1.1 Let $\mathcal{P}=\bigcup_{n\in N}\mathcal{P}_n$ be a family of subsets of a space X, here each $\mathcal{P}_n$ is a cover of X.

(1) $\{\mathcal{P}_n\}$ is a point-star network if $<st(x, \mathcal{P}_n)>$ is a network of x in X for each $x\in X$.

(2) $\{\mathcal{P}_n\}$ is a C point-star network (or, point-star network of C) if X has a point-star network $\{\mathcal{P}_n\}$ such that each $\mathcal{P}_n$ has the property C.

(3) $\{\mathcal{P}_n\}$ is a point-star D-network if X has a point-star network $\{\mathcal{P}_n\}$ such that each $<st(x, \mathcal{P}_n)>$ is a D-network of x in X.

Definition 3.1.4 Let $\mathcal{P}$ is a cover of a space X.

(1)(Li[2000a]) $\mathcal{P}$ is a cs*-cover if whenever S is a convergent sequence of X there is $P\in\mathcal{P}$ such that a subsequence of S is eventually P.

(2)(Yan[1998]) $\mathcal{P}$ is a cs-cover if each convergent sequence of X is eventually some element of $\mathcal{P}$.

(3)(Lin-Yan[2001b]) $\mathcal{P}$ is a sn-cover if each element of $\mathcal{P}$ is a sequential neighborhood of a point in X and there is a sequential neighborhood $P\in\mathcal{P}$ of x in X for each $x\in X$.

Theorem 3.1.6 (1) A space X is a sequentially quotient(or, subsequence-covering) π-image of a metric space iff X has a point-star network of cs*-covers.

(2) A space X is a 2-sequence-covering π-image of a metric space iff X has a point-star network of so-covers.

Theorem 3.1.7 The following are equivalent for a space X:

(1) X is a 1-sequence-covering π-image of a metric space.

(2) X is a sequence-covering π-image of a metric space.

(3) X has a point-star network of sn-covers.

(4) X has a point-star network of cs-covers.

Open question 3.1.14 Is a space with a point-star network of cs*-covers a pseudo-sequence-covering π-image of a metric space?

Open question 3.1.15(Tanaka[1987a]) Let X be a symmetric space. Is X a semimetric space if X contains no closed copy of S_2 ?

3.2 Point-finite point-star networks and compact mappings

In this section we discuss some relations between point-finite point-star networks and sequence-covering compact images of metric spaces

Theorem 3.2.2(Ikeda-Liu-Tanaka[2001], Yan[1997]) The following are equivalent for a space X:

(1) X is a pseudo-sequence-covering(or, sequentially quotient, subsequence-covering) and compact image of a metric space.

(2) X has a point-finite point-star network of cs*-covers.

(3) X has a point-finite point-star sn-network.

(4) X has a point-finite refined point-star cs*-network.

Theorem 3.2.4(Yan[1997]) A space X is a compact-covering and compact image of a metric space iff X has a point-finite point-star network of compact-finite-partitions.

Open question 3.2.7(Ikeda-Liu-Tanaka[2001]) Find a nice internal characterization of the quotient and compact images of locally separable metric spaces.

Theorem 3.2.8(Ikeda-Liu-Tanaka[2001], Lin-Yan[2001b], Yan[1998]) The following are equivalent for a space X:

(1) X is a 1-sequence-covering and compact image of a metric space.

(2) X is a sequence-covering and compact image of a metric space.

(3) X has a point-finite point-star network of sn-covers.

(4) X has a point-finite point-star network of cs-covers.

Open question 3.2.11(Ikeda-Liu-Tanaka[2001]) Is a symmetric space with a σ-point-finite cs-network a quotient and compact image of a metric space?

Open question 3.2.12 Is a quotient and π-image of a separable metric space a quotient and compact image of a separable metric space?

3.3 Point-regular covers

In this section we characterize sequence-covering compact images of metric spaces and compact-covering compact images of separable metric spaces by point-regular covers and sn-networks.

Definition 3.3.1(Alexandroff[1960]) Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is point-regular if $x \in U \in \tau$, then $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite.

(2) $\mathcal{P}$ is uniform if $x \in X$ and $\mathcal{P}$' is an infinite subset of $(\mathcal{P})_x$, then $\mathcal{P}$' is a network of x in X.

(3) $\mathcal{P}$ is regular if $x \in U \in \tau$, then there is a neighborhood V of x in X such that $\{P \in (\mathcal{P})_V : P \not\subset U\}$ is finite.

Question 3.3.2(Ikeda-Liu-Tanaka[2001]) (1) Find a nice mapping such that a sequential space with a point-regular cs-network can be characterized by the image of metric spaces under the mappings.

(2) Find a nice mapping such that a sequential space with a point-regular cs*-network can be characterized by the image of metric spaces under the mappings.

Theorem 3.3.8(Lin-Yan[2001b]) The following are equivalent for a space X:

(1) X is a 1-sequence-covering and compact image of a metric space.

(2) X is a sequence-covering and compact image of a metric space.

(3) X has a point-regular cs-network.

(4) X has a point-regular sn-network.

(5) X has a uniform cs-network.

(6) X has a uniform sn-network.

(7) X has a point-finite point-star network of cs-covers.

(8) X has a point-finite point-star network of sn-covers.

Theorem 3.3.12 A space X is a pseudo-sequence-covering(or, sequentially quotient, subsequence-covering) and s, π-image of a metric space iff X has a point-countable point-star network of cs*-covers.

Theorem 3.3.17 A regular and k-space with a regular k-network is metrizable.

Open question 3.3.19(Tanaka-Zhou[1985/86]) Does a quotient and compact image of a (locally compact) metric space has a point-G_δ property?

Open question 3.3.20 (1) Does a pseudo-sequence-covering and s, π-image of a metric space has a point-regular cs*-network?

(2) Is a space with a point-regular cs*-network a pseudo-sequence-covering and compact image of a metric space?

3.4 Sequence-covering mappings and 1-sequence-covering mappings

In this section we discuss how a sequence-covering mapping is 1-sequence-covering.

Question 3.4.1(Lin-Yan[2001a]) Is a Fréchet space with a countable cs-network a sequence-covering and closed image of a separable metric space?

Theorem 3.4.2(Lin-Yan[2001a]) Let $f:X\rightarrow Y$ be a sequence-covering and compact mapping. If X is metric, then f is 1-sequence-covering.

Open question 3.4.3 Is a sequence-covering and π-mapping of a metric space a 1-sequence-covering mapping?

Theorem 3.4.4(Yan-Lin-Jiang[2000]) Metrizability is preserved by sequence-covering and closed mappings.

Open question 3.4.5 Is g-metrizability preserved by sequence-covering and closed mappings?

Theorem 3.4.6(Yan-Lin-Jiang[2000]) Let $f:X\rightarrow Y$ be a sequence-covering and closed mapping. If X is metric, then f is 1-sequence-covering.

Open question 3.4.8 Let $f:X\rightarrow Y$ be a sequentially quotient and compact mapping. Is f pseudo-sequence-covering if X is metric?

Chapter 4 On hereditarily closure-preserving covers

In this chapter we discuss hereditarily closure-preserving covers and dominated families, establish the theory of closed mappings of metric spaces, study covering properties of g-metrizable spaces, and characterize certain Lašnev spaces

4.1 k-networks and covering properties

In this section we obtain paracompactness and metalindelöfness of some generalized metric spaces defined by k-networks.

Question 4.1.1(Siwiec[1974]) (1) Is a g-metrizable space a normal space?

(2) Does a separable g-metrizable space has a countable weak base?

(3) Is a normal g-metrizable space a paracompact space?

Theorem 4.1.4(Peng[2000]) A regular and k-space with a σ-hereditarily closure-preserving k-network is a hereditarily metalindelöf space.

Theorem 4.1.7(Liu[1995]) A normal and k-space with a σ-hereditarily closure-preserving k-network is a paracompact space.

Theorem 4.1.8(Gao[1992]) A normal space with a σ-closure-preserving weak base is a paracompact space.

Open question 4.1.10 Is a regular space with a σ-compact-finite weak base a metalindelöf space?

Open question 4.1.11 (1) Is a regular and k-space with a σ-locally finite k-network a space with a σ-closure-preserving weak base?

(2) Is a regular space with a σ-closure-preserving weak base a metalindelöf space?

4.2 Closed images of locally separable metric spaces

In this section we obtain some characterizations of closed images of locally separable metric spaces. In 5.1and 5.2 we shall introduce further results.

Theorem 4.2.6(Lin-Liu-Dai[1997]) The following are equivalent for a regular space X:

(1) X is a closed image of a locally separable metric space.

(2) X is a closed image of a metric space, and each first countable (closed) subspace of X is locally separable.

(3) X is a Fréchet space with a σ-hereditarily closure-preserving k-network by separable subspaces.

(4) X is a Fréchet space with a σ-hereditarily closure-preserving k-network by $\aleph_0$ -subspaces.

4.3 Dominated families and closed mappings

In this section we shall answer the following question posed by T. Miwa.

Question 4.3.1(Tanaka-Zhou[1985/86]) Let Y be a metric space. Is the Lašnev space Y/B dominated by a family of metric subspaces if B is a closed subspace of Y?

Theorem 4.3.3(Tanaka[1987a]) If a space X be dominated by a family of metric subspaces, then X is metrizable iff X contains no closed copy of S_2 and S_ω.

Theorem 4.3.5(Lin[1997b], Tanaka-Zhou[1985/86]) Let $f:Z \to X$ is a closed mapping. Then the following are equivalent for a metric space Z:

(1) X is dominated by a family of metric subspaces.

(2) X has a hereditarily closure-preserving closed covers by metric subspaces.

(3) X has a σ-hereditarily closure-preserving closed k-network by metric subspaces.

(4) f satisfies that

(a) each $\partial f^{-1}(x)$ is locally compact in Z.

(b) $\{x \in X: \partial f^{-1}(x)$ is not compact in $Z\}$ is discrete in X.

Theorem 4.3.7(Siwiec1976], Tanaka-Liu[1999]) Let $f:Z \to X$ be a closed mapping. Then the following are equivalent for a metric space Z:

(1) X is dominated by a countable family of metric subspaces.

(2) X has a σ-locally finite closed k-network by metric subspaces.

(3) X has a point-countable closed k-network by metric subspaces.

(4) X has a point-countable cs*-network by metric subspaces, and $\{x \in X: \partial f^{-1}(x)$ is not compact in $Z\}$ is discrete in X.

(5) f satisfies that

(a) each $\partial f^{-1}(x)$ is locally compact and Lindelöf in Z.

(b) $\{x \in X: \partial f^{-1}(x)$ is not compact in $Z\}$ is discrete in X.

Open question 4.3.8 Let $f:Z \to X$ be a closed mapping, and Z be a metric space. Characterize a space X such that X has (a) or (b) in Theorem 4.3.5, respectively.

Theorem 4.3.9(Lin[1997b]) The following are equivalent for a metric space Z:

(1) Each closed image of Z is dominated by a family of metric subspaces.

(2) If F is a closed subspace of Z, then ∂F is locally compact.

(3) The set of non-isolated points of Z is locally compact.

(4) Z is a locally normal metric space.

Chapter 5 On star-countable covers

In this chapter we study star-countable covers and the theory of quotient mappings of locally separable metric spaces, discuss some relations between spaces with a star-countable k-network and generalized metric spaces, and obtain some necessary and sufficient conditions of k-space properties of product spaces.

5.1 cs-networks and images of locally separable metric spaces

About quotient s-images of metric spaces, Velichko posed the following question.

Question 5.1.1(Velichko[1988]) Find a property Φ such that a Φ-space X is a quotient s-image of a metric space iff X is a quotient s-image of a Φ and metric space.

In this section we discuss some advances about the question.

Conjecture 5.1.3(Lin-Liu-Dai[1997]) A space X is a quotient s-image of a locally separable metric space iff X is a quotient s-image of a metric space, and each first countable subspace of X is locally separable.

Theorem 5.1.9(Lin-Yan[2001a], Tanaka-Xia[1996]) The following are equivalent for a space X:

(1) X is a sequence-covering s-image of a locally separable metric space.

(2) X has a point-countable cs-network by cosmic subspaces.

(3) X has a point-countable cs-network by $\aleph_0$-subspaces.

(4) X has a point-countable cs-network, and a so-covers by $\aleph_0$-subspaces.

Theorem 5.1.16(Lin[1995], Velichko[1988]) The following are equivalent for a regular and Fréchet space X:

(1) X has a locally countable cs-network.

(2) X has a locally countable k-network.

(3) X has a star-countable cs-network.

(4) X is a closed s-image of a locally separable metric space.

(5) X is a quotient s-image of a locally separable metric space.

(6) X is a locally separable space with a point-countable cs*-network.

(7) X has a point-countable cs*-network by separable subspaces.

Open question 5.1.19 Is a sequence-covering and compact image of a locally separable metric space equivalent to a space with a point-regular cs-network by $\aleph_0$-subspaces?

Open question 5.1.20 Let X be a regular and Fréchet space with a point-countable cs*-network. Is X locally separable if each first countable closed subspace of X is locally separable?

5.2 k-networks and Sakai's theorems

In this section we present some Sakai's work about spaces with a star-countable k-network and closed images of locally separable metric spaces.

Question 5.2.1(Ikeda-Tanaka[1993]) Is a regular and Fréchet space with a point-countable k-network by separable subspaces a closed image of a locally separable metric space?

Theorem 5.2.2(Sakai[1997a]) A regular and k-space X has a star-countable k-network iff X is dominated by a family of k and $\aleph_0$-subspaces.

Theorem 5.2.5(Liu-Tanaka[1998b], Sakai[1997a]) The following are equivalent for a regular and k-space X with a star-countable k-network:

(1) X is locally separable.

(2) X has a point-countable cs-network.

(3) X is a topological sum of $\aleph_0$-subspaces.

(4)(CH) χ (X)$\leq \omega_1$.

Theorem 5.2.8(Liu-Tanaka[1996c], Sakai[1997a]) The following are equivalent for a regular space X:

(1) X is a closed image of a locally separable metric space.

(2) X is a Fréchet space with a star-countable k-network.

(3) X is a Fréchet space with a point-countable k-network by separable subspaces.

Open question 5.2.10 Let X be a regular and Fréchet space with a point-countable k-network. Does X has a star-countable k-network if each first countable closed subspace of X is locally separable?

5.3 k-space properties of product spaces

Question 5.3.1(Michael[1973]) Find a necessary and sufficient condition such that X$\times$Y is a k-space for k-space X and Y.

In this section we discuss some new results of Michael's question on spaces with a point-countable k-network.

Definition 5.3.2 A sequence $\{X_i\}$ of spaces is called satisfied Tanaka's condition if one of the following conditions holds:

(1) All X_i are first countable.

(2) All X_i are k-spaces, and all X_i are compact expect for finite i, all X_i are locally compact expect for one i.

(3) All X_i are locally k_ω, and all X_i are compact expect for finite i.

If a sequence $\{X_i\}$ of spaces satisfies the Tanaka's condition, then $\prod_{i\in N} X_i$ is a k-space.

Conjecture 5.3.4 Let X and Y be k-spaces with a point-countable k-network. $X \times Y$ is a k-space iff the pair (X, Y) satisfies the Tanaka's condition.

Theorem 5.3.8(Liu-Lin[1997]) Let $X \times Y$ be a regular and k-space. Then the pair (X, Y) satisfies the Tanaka's condition if spaces X, Y satisfy one of the following conditions.

(1) A sequential space with a point-countable cs-network.

(2) A Fréchet space with a point-countable cs*-network.

(3) A space with a compact-countable cs*-network.

Theorem 5.3.14(Lin[1998b], Liu-Lin[1997], Liu-Tanaka[1996c]) The following are equivalent:

(1) $S_\omega \times S_{\omega_1}$ is not a k-space.

(2) If X, Y are regular and Fréchet spaces with a point-countable k-network, or regular spaces with a compact-countable k-network, then $X \times Y$ is a k-space iff the pair (X, Y) satisfies the Tanaka condition.

Open question 5.3.16(Liu-Tanaka[1998b]) Let X be a regular and Fréchet space with a point-countable k-network. Does X has a compact-countable k-network?

Theorem 5.3.19 Let $\prod_{i\in N} X_i$ be a regular k-space. Then the sequence $\{X_i\}$ of spaces satisfies the Tanaka's condition if each space X_i satisfies one of the following conditions.

(1) A sequential space with a point-countable cs-network.

(2) A Fréchet space with a point-countable cs*-network.

(3) A space with a compact-countable cs*-network.