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Spaces with a Locally Countable k-network

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Abstract In this paper, we consider the spaces with a locally countable k-network. It is shown that a regular k-space with a locally countable k-network is an $\aleph$-space, and that there exists a completely regular space with a locally countable k-network which is not an $\aleph$-space. By using the above result, we prove that a regular space is a Fréchet space with a locally countable k-network if and only if it is a pseudo-open (or closed) s-image of a locally separable metrizable space.

Key Words k-network; $\aleph$-space; k-space

A space is a locally separable metrizable space if and only if it is a regular space with a locally countable base (cf. [1]). Thus, one may investigate the further properties of locally separable metrizable spaces by means of the discussion of properties of spaces with a locally countable k-network. From the classical Nagata-Smirnov metrization theorem we know that a regular space with a locally countable base has a σ-locally finite base. So, the following question can be raised:

Question 1 Is a regular space with a locally countable k-network a space with a σ- locally finite k-network?

Since a space with a locally countable k-network is a generalization of a locally separable metrizable space, and since our purpose is to bring out properties of locally separable metrizable spaces by means of that of the space with a locally countable k-network, according to Alexandroff's hypothesis, the following question can be raised:

Question 2 By means of what maps can we establish the relationship between locally separable metrizable spaces and spaces with a locally countable k-network?

In this paper, we prove that a regular k-space with a locally countable k-network has a σ-locally finite k-network (Theorem 1) and that there exists a completely regular space with a locally countable k-network, which has no σ-locally finite k-network (Example 1). Secondly, by means of pseudo-open (or closed) s-maps we establish the relationship between locally separable metrizable spaces and spaces with a locally countable k-network, and prove that a regular space is a Fréchet space with a locally countable k-network if and only if it is a pseudo-open (or closed) s-image of a locally serapable metrizable space (Theorem 2).

In the following, all maps are continuous and surjective, and N denotes the set of positive integers.

Let X be a topological space. A collection $\mathscr{P}$ of subsets of X is called a k-network if for any com-

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pact subsets K of an open set V of X, there exists a finite subcollection $\mathscr{P}'$ of $\mathscr{P}$ such that

$$K \subset \bigcup \mathscr{P}' \subset V.$$

A regular space is called an $\aleph_0$-space if it has a countable k-network. A regular space is called an $\aleph$-space if it has σ-locally finite k-network. A subset V of a space X is said to be sequentially open if every sequence converging to a point of V is eventually in V. A space X is called a sequential space if every sequentially open subset of X is open in X. A space X is called a k-space if for each $A \subset X$, A is closed in X whenever the intersection of A with any compact subspace K of X is closed in K. Every sequential space is a k-space. A space X is called a meta-Lindelöf space if every open cover of X has a point countable open refinement.

Proposition 1 A regular sequential space with a σ-locally countable k-network is a hereditary meta-Lindelöf space.

Proof A space X is a hereditary meta-Lindelöf space if and only if every open subset of X is a meta-Lindelöf space. Since regularity is hereditary, and since the property that a sequential space X has a σ-locally countable k-network is hereditary to all open subspace of X, to complete the proof of the proposition it suffices to show that a regular sequential space with a σ-locally countable k-network is a meta-Lindelöf space. Suppose X is a regular sequential space with a σ-locally countable k-network. By regularity, we can assume that X has a σ-locally countable closed k-network $\bigcup_{n \in N} \mathscr{P}_n$, where each $\mathscr{P}_n$ is locally countable and $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. We assert that each locally countable collection of subsets of X may be expanded to a point countable collection of open sets of X. In fact, let

$$\mathscr{F} = \{F_a : a \in A\}$$

be a locally countable collection of subsets of X. Let

$$\Lambda = \{\lambda : \lambda \text{ is a finite sequence consisting of members of } N\}.$$

For each $\lambda \in \Lambda$, a locally countable family

$$\mathscr{F}(\lambda) = \{F_a(\lambda) : a \in A\}$$

is defined inductively as follows:

$$\mathscr{F}(\phi) = \mathscr{F}$$

where $F_a(\phi) = F_a$ for each $a \in A$, and if a locally countable family $\mathscr{F}(\lambda)$ has been defined and $n \in N$, then put

$$\mathscr{P}(\hat{\lambda n}) = \{P \in \mathscr{P}_n : P \cap F_a(\lambda) \neq \phi \text{ for only countablely many } a \in A\},$$

$$F_a(\hat{\lambda n}) = \bigcup \{P \in \mathscr{P}(\hat{\lambda n}) : P \cap F_a(\lambda) \neq \phi\},$$

and

$$\mathscr{F}(\hat{\lambda n}) = \{F_a(\hat{\lambda n}) : a \in A\}.$$

For each $x \in X$, since $\mathscr{P}_n$ is locally countable, there exists a neighbourhood V of x and a countable subcollection $\mathscr{P}'$ of $\mathscr{P}_n$ such that $P \cap V = \phi$ when $P \in \mathscr{P}_n - \mathscr{P}'$. Let

$$\mathscr{P}' \cap \mathscr{P}(\hat{\lambda n}) = \{P_i : i \in N\}.$$

For each $i \in N$, there exists a countable subset A_i of A such that

$$P_i \cap F_a(\lambda) = \phi \qquad \text{when } a \in A - A_i.$$

So $\{a \in A : \text{there exists } i \in N \text{ such that } F_a(\lambda) \cap P_i \neq \phi\}$ is countable, i.e.,

$$V \cap F_a(\widehat{\lambda n}) \neq \phi$$

for only countablely many $a \in A$. Therefore, $\mathscr{P}(\widehat{\lambda n})$ is locally countable. Now for each $a \in A$, put

$$W_a = \bigcup \{F_a(\lambda) : \lambda \in \Lambda\}.$$

If $\mathscr{W} = \{W_a : a \in A\}$ is not point countable, then there exists $x_0 \in X$ and an uncountable subset A' of A such that $x_0 \in W_a$ for each $a \in A'$. For each $a \in A'$, there exists $\lambda(x_0, a) \in \Lambda$ such that $x_0 \in F_a(\lambda(x_0, a))$. Since A' is uncountable, there is an uncountable subset A'' of A' and $\lambda_0 \in \Lambda$ with $\lambda(x_0, a) = \lambda_0$ where $a \in A''$, i. e. , $x_0 \in F_a(\lambda_0)$. This contradicts that $\mathscr{F}(\lambda_0)$ is point countable. Hence $\mathscr{W}$ is point countable. Obviously, $\mathscr{W}$ is an expansion of $\mathscr{F}$. To show that each W_a is open, it suffices to show that W_a is sequentially open because X is sequential. Suppose a sequence $Z = \{x_n\}$ converges to $x \in W_a$. Then there exists $\lambda \in \Lambda$ with $x \in F_a(\lambda)$. Since $\mathscr{F}(\lambda)$ is locally countable, there exists a neighbourhood V of x such that

$$V \cap F_b(\lambda) \neq \phi$$

for only countablely many $b \in A$. Since $x \in V$, there exists $n \in N$ such that

$$Z_n = \{x\} \cup \{x_m : m \geqslant n\} \subset V.$$

And hence, there exist $m, h \in N$ and $P_i \in \mathscr{P}_m$ $(i \leqslant h)$ such that

$$Z_n \subset \bigcup_{i \leqslant h} P_i \subset V.$$

We can assume that there exists $h_1 \leqslant h$ with $x \in (\bigcap_{i \leqslant h_1} P_i) - (\bigcup_{h_1 < i \leqslant h} P_i)$. Since the P_i's are closed, $x_n \in \bigcup_{h_1 < i \leqslant h} P_i$ for only finitely many $n \in N$. Therefore, the sequence Z is eventually in $\bigcup_{i \leqslant h_1} P_i$. Since $P_i \subset V$ and $x \in P_i \cap F_a(\lambda)$ for each $i \leqslant h_1$,

$$\bigcup_{i \leqslant h_1} P_i \subset F_a(\widehat{\lambda m}) \subset W_a$$

by the definition of $\mathscr{P}(\widehat{\lambda m})$. Hence Z is eventually in W_a, and $\mathscr{W}$ is a point countable open expansion of $\mathscr{F}$.

Let $\mathscr{U}$ be an open cover of X. Since X has a σ-locally countabel k-network, there exists a locally countable collection $\mathscr{F}_i = \{F_{i,a} : a \in A_i\}$ of subsets of X such that $\bigcup_{i \in N} \mathscr{F}_i$ is a refinement of $\mathscr{U}$. For each $i \in N$, there exists a point countable open expansion

$$\mathscr{W}_i = \{W_{i,a} : a \in A_i\}$$

of $\mathscr{F}_i$ with $F_{i,a} \subset W_{i,a}$. For each $a \in A_i$, we can choose $U_{i,a} \in \mathscr{U}$ with $F_{i,a} \subset U_{i,a}$. Then

$$\bigcup_{i \in N} \{U_{i,a} \cap W_{i,a} : a \in A_i\}$$

is a point countable open refinement of $\mathscr{U}$. Hence X is a meta-Lindelöf space.

Theorem 1 A regular k-space with a locally countable k-network is a topological sum of $\aleph_0$-spaces, and hence it is an $\aleph$-space.

Proof Suppose a topological space X is a regular k-space with a locally countable k-network. Since X is a local $\aleph_0$-space, X is a locally separable space with point G_δ-property (cf. [2], (D)). By Proposition 1, X is a hereditary meta-Lindelöf space because a regular k-space with point G_δ-property is sequential (cf. [3], Theorem 7. 3). So X is a locally separable and hereditary meta-Lindelöf space and hence it is a topological sum of Lindelöf spaces (cf. [4], Proposition 8. 7). Since every locally countable collection of subsets of a Lindelöf space is countable, X is a topological sum of $\aleph_0$-spaces.

Therefore, X is an $\aleph$-space because $\aleph_0$-spaces are $\aleph$-spaces and $\aleph$-properties are preserved for topological sums.

Example 1 There exists a completely regular space with a locally countable k-network, which is not an $\aleph$-space.

Proof Let

$$X = \omega_1 \cup (\omega_1 \times \{1/n; n \in N\}),$$

and define a base $\mathscr{B}$ for the desired topology on X as follows:

(1) if $x \in X - \omega_1$, let $\{x\} \in \mathscr{B}$, and

(2) if $a \in \omega_1$, then

$$\{\{a\} \cup (\bigcup_{n \geq m} V(n,a) \times \{1/n\}) : m \in N, V(n, a) \text{ is a neighbourhood of } a \text{ in } \omega_1 \text{ with the order topology}\} \subset \mathscr{B}.$$

Since X has an open and closed base, X is a completely regular space. Put

$$\mathscr{P} = \{\{x\}: x \in X\} \cup \{\{a\} \cup \{(a, 1/n): n \geq m\}: m \in N, a \in \omega_1\}.$$

Then $\mathscr{P}$ is a locally countable collection of subsets of X. If K is a compact subset of X, then

(a) $K \cap \omega_1$ and $K \cap (\omega_1 \times \{1/n\})$ are finite for each $n \in N$;

(b) $a \in \omega_1 - K$, implies $(a, 1/n) \in K$ for only finitely many $n \in N$;

(c) $K - \bigcup \{\{a\} \cup \{(a, 1/n): n \in N\}: a \in K \cap \omega_1\}$ is finite.

(a) holds because ω_1 and $\omega_1 \times \{1/n\}$ are closed discrete subspaces of X for each $n \in N$.

For each $a \in \omega_1 - K$, suppose that there exists countablely many $n \in N$, say $\{n_i\}$, such that $(a, 1/n_i) \in K$. Since K is compact, $\{(a, 1/n_i): i \in N\}$ has a cluster point $x \in K$. Then

$$x = a \in \omega_1 - K,$$

a contradiction. Hence (b) holds.

By the same reason, (c) holds.

It is not difficult to check that $\mathscr{P}$ is a locally countable k-network for X.

To show that X is not an $\aleph$-space, we first prove the following lemma.

Lemma A pseudo-open and compact image of a K-semistratifiable space is a semistratifiable space.

Proof By the definition of a K-semistratifiable space (cf. [5]), a topological space X is a K-semistratifiable space if and only if X has a K-semistratification, i. e., to each closed subset F of X one can assign a sequence $\{G(F, n)\}_{n \in N}$ of open subsets of X such that

(a) $G(F, n+1) \subset G(F, n)$ for each $n \in N$;

(b) $F = \bigcap_{n \in N} G(F, n)$;

(c) $G(F, n) \subset G(H, n)$ whenever $F \subset H$ and $n \in N$;

(d) if K is a compact subset of X and $K \cap F = \phi$, there exists $n \in N$ such that

$$K \cap G(F, n) = \phi.$$

Let f be a pseudo-open and compact map from the K-semistratifiable space X onto a topological space Y, to show that Y is a semistratifiable space, it suffices to show that Y has a semistratification (cf. [6]), i. e., to each closed subset F of Y one can assign a sequence $\{S(F, n)\}_{n \in N}$ of open subsets of Y satistying (a)—(c). Let F be a closed subset of Y. Then $f^{-1}(F)$ is closed in X and, put

$$S(F, n) = \text{int}(f(G(f^{-1}(F), n))),$$

we will show that the correspondence $F \to \{S(F, n)\}_{n \in N}$ is a semistratification for Y. (a) and (c) are easily shown to be satisfied. Since f is a pseudo-open map, $F \subset S(F, n)$ for each $n \in N$, and hence

$$F \subset \bigcap_{n \in N} S(F, n).$$

On the order hand, if $y \in Y - F$, then

$$f^{-1}(y) \cap f^{-1}(F) = \phi.$$

Since $f^{-1}(y)$ is a compact subset of X, there exists $m \in N$ such that

$$f^{-1}(y) \cap G(f^{-1}(F), m) = \phi,$$

i. e. , $y \overline{\in} f(G(f^{-1}(F), m))$. But

$$S(F, m) \subset f(G(f^{-1}((F), m)),$$

therefore $y \overline{\in} \bigcap_{n \in N} S(F, n)$. Hence

$$F = \bigcap_{n \in N} S(F, n),$$

(b) holds, and Y is a semistratifiable space. This completes the proof of the Lemma.

Since $\aleph$-spaces are K-semistreatifiable spaces (cf. [7]), to complete the proof of Example 1 it suffices to show X is not a K-semistratifiable spaces. Define $f: X \to \omega_1$ such that

$$f((a, 1/n)) = f(a) = a \qquad \text{for each } a \in \omega_1.$$

It is easily shown that f is a pseudo-open and compact map from X onto the space ω_1 with the order topology. But, ω_1 is not subparacompact because every subparacompact, countably compact space is compact. Since every semistratifiable space is a subparacompact space (cf. [6]), ω_1 is not a semistratifiable space. By the above Lemma, X is not a K-semistratifiable space. Hence X is not an $\aleph$-space. This completes the proof of Example 1.

In the second part of this paper, we will establish the relationship between locally separable metrizable spaces and spaces with a locally countable k-network by means of suitable maps. A map $f: X \to Y$ is an s-map if for each $y \in Y$, $f^{-1}(y)$ is separable in X. A space X is a Fréchet space if for every $A \subset X$ and every $x \in \text{cl}(A)$, there exists a sequence of points of A converging to x. Every first countable space is a Frécher space and every Fréchet space is a k-space.

Theorem 2 The following are equivalent for a regular space X:

(1) X is a closed s-image of a locally separable metrizable space.

(2) X is a pseudo-open s-image of a locally separable metrizable space.

(3) X is a Fréchet space with a locally countable k-network.

Proof Since every closed map is pseudo-open, (1)$\Rightarrow$(2) is obvious. Suppose a regular space X is a pseudo-open s-image of a locally separable metrizable space. Since every locally separable metrizable space is a regular, locally separable, Fréchet space with a point countable k-network, X is a topological sum of $\aleph_0$-seaces (cf. [4], Proposition 8. 8). So X has a locally countable k-network. X is a Fréchet space, because Fréchet property is preserved under pseudo-open maps. Hence (2)$\Rightarrow$(3). Finally, suppose X is a Fréchet space with a locally countable k-network. Since every Fréchet space is a k-space, X is a topological sum of $\aleph_0$-spaces by Theorem 1. Let $X = \bigoplus_{a \in A} X_a$, where each X_a is an $\aleph_0$-space. For each $a \in A$, since Fréchet property is hereditary, X_a is a Fréchet $\aleph_0$-space. So X_a is a Laśnev space (cf. [7], Corollary 9), i. e. , X_a is the image of a metrizable space M_a under a closed map f_a. Since metrizable space M_a is paracompact, and X_a is Fréchet, there exists a closed subset M'_a of M_a such that $f_a(M'_a)=X_a$ and $f_{a|M'_a}: M'_a \to X_a$ is an irreducible map, i. e. , if H_a is a closed subset of M'_a such that $f_{a|M'_a}(H_a)=X_a$, then $H_a=M'_a$ (cf. [8], Theorem 55. 12). Without loss of generality, we can assume that f_a is a closed irreducible map. Since X_a is an $\aleph_0$-space, it is separable. Let A_a be a countable dense subset of X_a. Take a countable subset B_a of M_a with $f_a(B_a)=A_a$. Then

$$f_a(\mathrm{cl}(B_a)) = \mathrm{cl}(f_a(B_a)) = \mathrm{cl}(A_a) = X_a.$$

Since f_a is irreducible, $\mathrm{cl}(B_a)=M_a$. So M_a is a separable metrizable space. Let $M=\bigoplus_{a\in A} M_a$ and define $f: M\to X$ by $f|_{M_a}=f_a$ for each $a\in A$. Then f is a closed map from the locally separable metrizable space M onto X, and for each $x\in A$, there exists $a\in A$ with $f^{-1}(x)\subset M_a$. Hence $f^{-1}(x)$ is a separable subspace of M. Therefore, f is a closed s-map. This completes the proof of the Theorem.

Remark A regular quotient s-image of a locally separable metrizable space need not be a space with a locally countable k-network (cf. [9], Example 4).

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