Math. Japonica **46**, No. 3(1997), 377–384

ON SPACES WITH A $\sigma - CF^*$ PSEUDO–BASE

TAKEMI MIZOKAMI AND SHOU LIN

Received September 21, 1995; revised December 18, 1995

ABSTRACT. We propse the conjecture that a space has a $\sigma - CF^*$ pseudo-base if and only if X is either an $\aleph_0$-space or has the property that all compact subsets are finite, and give an almost positive answer to it.

1. Introduction. All spaces are assumed to be regular T_2-spaces. The letter N always denotes the set of natural numbers. For a space X, let $\mathcal{K}(X)$ be a family of all non-empty compact subsets of X and let $\tau(X)$ be the topology of X.

A family $\mathcal{P}$ of subsets of X is called a *pseudo-base* (briefly *p.b.*) *for* X if for each pair K, U of $K \in \mathcal{K}(X)$ and $U \in \tau(X)$ with $K \subset U$ there exists $P \in \mathcal{P}$ such that $K \subset P \subset U$. A space X is called an $\aleph_0$-*space* if it has a countable p.b. [4]. The second author established the following characterization of a space having a σ-*HCP* (*hereditarily closure-preserving*) p.b.: A space has a σ-HCP p.b. if and only if either X is an $\aleph_0$-space or $X = \cup\{X_i : i \in N\}$, where each X_i is a closed discrete subset, and X has the property ACF (=All compact subsets are finite). On the other hand, the first author generalized both compact-finite and HCP families to CF, CF^* families such as: A family $\mathcal{U}$ of subsets of a space X is called CF *in* X if $\mathcal{U}|K$ (the family of the intersections of members of $\mathcal{U}$ with K) $= \{U_1, \cdots, U_k\}$ is finite for each $K \in \mathcal{K}(X)$, and called CF^* *in* X if additionally the following is satisfied: If $|U_i| \geq \aleph_0$, then $\{U \in \mathcal{U} : U \cap K = U_i\}$ is finite [5, Definition 3.1 and 3.6]. Taking the implication

$$HCP \to CF^* \to CF$$

into account, it is natural to propose the following conjecture:

(A) A space X has a σ-CF^* p.b. if and only if either X is an $\aleph_0$-space or X has the property ACF.

We note that since an $\aleph_0$-space or a space with the property ACF has a σ-CF^* p.b., this conjecture is essentially the truth of the only if part. Our object here is to give an almost positive answer.

2. Spaces with a σ-CF^* p.b.. Before the discussion, let us note that any space is the quotient space of a space with a σ-CF^* p.b.. In fact, any space is the quotient space of a space with the property ACF [7]. A space is called *perfect* if each closed subset is a G_δ-set. The next gives an almost positive answer to (A).

Theorem 1. *A space X has a σ-CF^* p.b. consisting of perfect subsets of X if and only if either X is an $\aleph_0$-space or X has the property ACF.*

Key words and phrases. pseudo-base, CF^* family, Fréchet space, k-space.

Proof. If part is trivial. Only if part: Assume the negation of ACF, i.e., that there exists an infinite compact subset K of X. Let $\mathcal{P}$ be a σ-CF^* p.b. consisting of perfect subsets of X. Let

$$\mathcal{P}(K) = \{P \in \mathcal{P} : K \subset P\}.$$

Then $\mathcal{P}(K)$ is countable. We make the following additional assumption, under which we show that X is an $\aleph_0$-space.

Case (a): There exists another infinite compact subset L of X such that $K \cap L = \phi$. Obviously

$$\mathcal{P}(L) = \{P \in \mathcal{P} : L \subset P\}$$

is countable. Then it follows that $\mathcal{P}(K) \wedge \mathcal{P}(L)$ forms a countable p.b. for X. Indeed, let $C \subset U$ with $C \in \mathcal{K}(X)$ and $U \in \tau(X)$. Take disjoint $U_1, U_2 \in \tau(X)$ such that $K \subset U_1$ and $L \subset U_2$. Then there exist $P_1 \in \mathcal{P}(K)$ and $P_2 \in \mathcal{P}(L)$ such that

$$K \cup C \subset P_1 \subset U_1 \cup U \quad \text{and} \quad L \cup C \subset P_2 \subset U_2 \cup U,$$

which implies $C \subset P_1 \cap P_2 \subset U$.

Case (b): Let K be a G_δ-set of X. Note that K is metrizable because K has a G_δ-diagonal. Then easily we can choose a countably family $\mathcal{W}$ of open subsets of X closed under finite intersections satisfying the following: If K' is a compact subset of K, then we have $K' = \cap\{\bar{W} : W \in \mathcal{W}(K')\}$, where

$$\mathcal{W}(K') = \{W \in \mathcal{W} : K' \subset W\}.$$

In this case, we show that

$$\mathcal{Q} = \{D \setminus W : D \in \mathcal{P}(K), W \in \mathcal{W}\} \cup \mathcal{P}(K)$$

forms a p.b. for X. Let $C \subset U$ with $C \in \mathcal{K}(X)$ and $U \in \tau(X)$. If $K \subset U$, then obviously $K \cup C \subset P \subset U$ for some $P \in \mathcal{P}(K)$. If $K' = K \setminus U \neq \phi$, then by the property of $\mathcal{W}$ above, there exists $W \in \mathcal{W}(K')$ such that $K' \subset W \subset X \setminus C$. Take $P \in \mathcal{P}(K)$ such that

$$K \cup C \subset P \subset U \cup W$$

Then easily we have $C \subset P \setminus W \subset U$.

Assume the negation of both (a) and (b), under which we show a contradiction. Let $\mathcal{P}^*(K)$ be the totality of finite intersections of members of $\mathcal{P}(K)$. Let $x \notin K$ be fixed for a while. Take disjoint open subsets $U(x), V(x)$ of X such that $x \in U(x)$ and $K \subset V(x)$. We settle the following claim:

Claim: *There exists $P(x) \in \mathcal{P}^*(K)$ such that*

$$\{x\} \cup K \subset P(x) \subset U(x) \cup V(x)$$

and

$$U(x) \cap P(x) = \{x\}.$$

The proof of claim: Otherwise, it follows that $U(x) \cap P(x)$ is infinite for each $P \in \mathcal{P}^*(K)$ such that

$$\{x\} \cup K \subset P \subset U(x) \cup V(x).$$

Choose $\{P_n : n \in N\} \subset \mathcal{P}(K)$ such that

$$\{x\} \cup K \subset P_{n+1} \subset P_n \subset U(x) \cup V(x)$$

for each n and such that if $\{x\} \cup K \subset U \in \tau(X)$, then $\{x\} \cup K \subset P_n \subset U$ for some n. Since $U(x) \cap P_n$ is infinite for each n, we can choose a sequence $\{x_n : n \in N\}$ of points such that

$$x_n \in (P_n \setminus \{x\}) \cap U(x) \setminus \{x_1, \cdots, x_{n-1}\}$$

for each n. This implies

$$L = \{x_n : n \in N\} \cup \{x\} \in \mathcal{K}(X)$$

and $L \cap K = \phi$, which implies (a). Thus the claim is settled. By the nagation of (b), $X \setminus K$ is uncountable. Since $\mathcal{P}^*(K)$ is countable, there exists $P_0 \in \mathcal{P}^*(K)$ such that

(*) $$D = \{x \in X \setminus K : P(x) = P_0\}$$

is an uncountable discrete subset of P_0. By the perfectness of P_0, there exists an uncountable closed discrete subset D_0 of P_0. Since $\mathcal{P}$ is a p.b. for X, for each $F \in \mathcal{F}(D_0)$ (= the totality of non-empty finite subsets of D_0) we can take $P(F) \in \mathcal{P}^*(K)$ such that $K \cup F \subset P(F)$ and $P(F) \cap D_0 = F$. But this is a contradiction because $\{P(F) : F \in \mathcal{F}(D_0)\}$ is uncountable. This completes the proof.

According to Sakai's example [7], a space with a σ-CF* p.b. need not be perfect. In fact, there he constructed a space X with the property ACF but X is not countably metacompact. A space X is called $\aleph_1$-*compact* if every closed discrete subspace of X is countable [2], and X is called σ-*discrete*, σ-*closed discrete* if $X = \cup\{X_n : n \in N\}$ where each X_n is discrete, closed discrete in X, respectively.

Corollary 1. *If a space X has a σ-CF^* p.b., then one of the following three cases holds:*
(1) *X is an $\aleph_0$-space.*
(2) *X has the property ACF.*
(3) *X is an $\aleph_1$-compact, σ-discrete space.*

Proof. We show that under the negation of both (1) and (2), (3) is true. We repeat the same discussion as above. By virtue of the claim, for each $x \notin K$ there exist $P(x) \in \mathcal{P}^*(K)$ and an open neighborhood $U(x)$ of x in X such that

$$K \cup \{x\} \subset P(x) \quad \text{and} \quad U(x) \cap P(x) = \{x\}.$$

For each $P \in \mathcal{P}^*(K)$, set

$$D(P) = \{x \in X \setminus K : P(x) = P\}.$$

Then each $D(P)$ is a discrete subset of X. We note that in the previous proof we can assume that K is a convergent sequence with its limit point. Hence we have

$$X = \cup\{D(P) : P \in \mathcal{P}^*(K)\} \cup K$$

is the union of countably many discrete subsets of X. If X is not $\aleph_1$-compact, then there exists an uncountable closed discrete subset D of X. By the repetition of the last part below (*) in the proof of Theorem 1, we have a contradiction. Hence X is $\aleph_1$-compact.

Corollary 2. *Our conjecture (A) is true if the following conjecture (B) is true: (B) Let* $X = K \cup D$, *where* K *is an infinite compact subset and* D *is an uncountable discrete subset of* X *such that* $K \cap D = \phi$ *and let* X *be* $\aleph_1$*-compact. Then* X *has no* σ*-CF* p.b..*

Proof. Assume the validity of (B) and X has a σ-CF^* p.b.. Then by the proof of Theorem 1, one of the following three cases holds: (1) X is an $\aleph_1$-space, (2) X has the property ACF and (3) there exists an uncountable discrete subset D and an infinite compact subset K of X such that $K \cap D = \phi$. By the same way as the proof of Corollary 1, we can show that the subspace $K \cup D$ is an $\aleph_1$-compact space. Since σ-CF^* p.b.s are hereditary to any subspace of $X, K \cup D$ has a σ-CF^* p.b.. But this is a contradiction to (B). Hence either (1) or (2) holds, implying the validity of (A).

Theorem 2. *If a space* X *has a* σ*-CF* p.b., then* X *is Lindelöf or* X *has the property ACF.*

Proof. Let $\mathcal{P}$ be a σ-CF^* p.b. for X. Assume that X has an infinite compact subset of K. It suffices to show that X is Lindelöf. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, there exist $P(x) \in \mathcal{P}$ and a finite subfamily $\mathcal{U}(x)$ of $\mathcal{U}$ such that

$$\{x\} \cup K \subset P(x) \subset \cup\mathcal{U}(x).$$

Note that $\mathcal{P}_0 = \{P(x) : x \in X\}$ is countable. For each $P \in \mathcal{P}_0$, let

$$X(P) = \{x \in X : P(x) = P\}$$

and take a finite subfamily $\mathcal{U}(P)$ of $\mathcal{U}$ such that $X(P) \subset \cup\mathcal{U}(P)$. Then $\cup\{\mathcal{U}(P) : P \in \mathcal{P}_0\}$ is a countable subcover of $\mathcal{U}$, proving that X is Lindelöf.

A space X is called a *k-space* if $F \subset X$ is closed if and only if $F \cap K$ is closed in K for each $K \in \mathcal{K}(X)$. In the class of k-spaces, conjecture (A) is true. More strictly, we have the following characterization:

Theorem 3. *Let* X *be a k-space. Then* X *has a* σ*-CF* p.b. if and only if* X *is either an* $\aleph_0$*-space or a discrete space.*

Proof. If part is trivial. Only if part: Assume that X is not a discrete space. Then there exists an infinite compact subset because of k-ness of X. By virtue of Theorem 2, X is Lindelöf, hence $\aleph_1$-compact. On the other hand, it is easily seen that a CF^* family in a k-space is $WHCP$ in it in the sense of [2]. Therefore X has a σ-$WHCP$ p.b. By virtue of [2, Lemma 1] X is an $\aleph_0$-space.

A space X is called a *Frécet space* if whenever $x \in \bar{A}$ there exists a sequence in A converging to x. A Fréchet space is a k-space. Under Fréchet spaces, we have a more stronger characterization of spaces with a σ-CF^* p.b.. For that matter, we prepare the characterization of Fréchet $\aleph_0$-spaces.

Lemma. *For a space* X*, TFAE:*
(1) X *is a Fréchet* $\aleph_0$*-space.*
(2) X *is a closed image of a subspace* M *of a Cantor set.*
(3) X *is a closed image of a separable metric space.*

Proof. (2) $\to$ (3) $\to$ (1) is trivial. We show (1) $\to$ (2). Let $\mathcal{P} = \{P(n) : n \in N\}$ be p.b. for X such that each $P(n)$ is closed in N. Without loss of generality we can assume that $\mathcal{P}$ is closed under finite unions. For each n, let

$$\Delta(n) = \{P(n), X \setminus P(n)\}$$

be a discrete space. Define a subspace M of $\Pi\{\Delta(n) : n \in N\}$ and a mapping $f : M \to X$ as follows:

$$M = \{\langle Q(n)\rangle \in \Pi\Delta(n) : \{\overline{Q(n)} \text{ is a local network at some point of } X\},$$
$$f(\langle Q(n)\rangle) = \cap\overline{Q(n)} \quad \text{for each } \langle Q(n)\rangle \in M.$$

For the latter use, we notice here the following claim the proof of which is the same as in Foged's one [1, Lemma 4].

Claim: *let $x \in U \in \tau(X)$ and Z be a convergent sequence of points of U such that $Z \to x$. Then there exists $P(n) \in \mathcal{P}$ such that $P(n) \subset U$ and Z is eventually in* $\mathrm{Int}P(n)$.

With the aid of the claim, we show that f has the required properties. To see that f is onto, let $x \in X$. If x is isolated, then it is easy to find $\langle Q(n)\rangle \in M$ with $f(\langle Q(n)\rangle) = x$. Let x be not isolated. By Fréchetness of X, there exists a sequence Z of points of $X \setminus \{x\}$ with $Z \to x$. For each n, we choose $Q(n) \in \Delta(n)$ such that $\overline{Q(n)} \cap Z$ is infinite. To see that $\{\overline{Q(n)}\}$ is a local network at x in X, let $x \in U \in \tau(X)$. By the claim above, there exists $n \in N$ such that $P(n) \subset U$ and Z is eventually in $\mathrm{Int}P(n)$. This implies $Q(n) = P(n)$ and $x \in \overline{Q(n)} \subset U$. Hence we have $f(\langle Q(n)\rangle) = x$. The continuity of f follows easily from the definition of M. Finally, we show that f is closed. Let A be a closed subset of M and suppose $p \in \overline{f(A)} \setminus f(A)$. There exists two sequences $\{P_n\}$ and $\{\langle Q_n(k)\rangle : n \in N\}$ satisfying $\langle Q_n(k)\rangle \in A, n \in N$ and

$$f(\langle Q_n(k)\rangle) = P_n \to p \text{ as } n \to \infty.$$

By induction we choose $\langle Q(n)\rangle \in M$ as follows: Choose $Q(1) \in \Delta(1)$ such that $N_1 = \{n \in N : Q_n(1) = Q(1)\}$ is infinite. Then we notice that $\overline{Q(1)}$ contains $\{p_n : n \in N_1\}$. By the same way we choose $Q(1) \in \Delta(2)$ such that $N_2 = \{i \in N_1 : Q_n(2) = Q(2)\}$ is infinite. We continue this process. By the same argument as above, we have $\langle Q(n)\rangle \in M$ and $f(\langle Q(n)\rangle) = p$. Since $\langle Q(n)\rangle \in \overline{\{\langle Q_n(k)\rangle : n \in N\}}$, $\langle Q(n)\rangle \in \bar{A} \subset A$, which implies $p \in f(A)$. But this is a contradiction.

Theorem 4. *For a space X, TFAE:*
(1) *X is a Fréchet space with a σ-CF^* p.b..*
(2) *X is a Fréchet space with a σ-HCP p.b..*
(3) *X is either a discrete space or a closed image of a subspace of a Cantor set.*
(4) *X is either a discrete space or a closed image of a separable metric space.*

Proof. (1) $\leftrightarrows$ (2) follows from [5, Proposition 3.8]. (1) $\to$ (3) $\to$ (4) $\to$ (1) follows from the previous lemma and Theorem 3.

3. The hyperspaces and σ-CF^* p.b.s. For a space X, let $\mathcal{K}(X)$ be topologized with the finite topology. That is, $\mathcal{K}(X)$ has a base consisting of all subsets of the following form:

$$\langle U_1, \cdots, U_k\rangle = \{K \in \mathcal{K}(X) : K \subset \cup\{U_i : 1 \le i \le k\} \text{ and } K \cap U_i \neq \phi \text{ for each } i,$$

where $\{U_i, \cdots, U_k\}$ is a finite family of open subsets of X. It is known that regular T_2-spaces are inherited to $\mathcal{K}(X)$ [3] and also that $\aleph_0$-spaces are so [4]. This is used in the next theorem.

Theorem 5. *Let X be a space. Then*

(1) *X has a σ-HCP p.b. if and only if so does $\mathcal{K}(X)$.*
(2) *X has a σ-CF^* p.b. consisting of perfect subsets of X if and only if so does $\mathcal{K}(X)$.*

Proof. If parts of both (1) and (2) are easily seen because X is homeomorphic to the closed subspace of $\mathcal{K}(X)$. Only if part of (1): Let X have a σ-HCP p.b.. Then by the characterization due to Lin stated in the introduction, X is either an $\aleph_0$-space or a σ-closed discrete space with the property ACF. If X is an $\aleph_0$-space, then so is $\mathcal{K}(X)$ [4]. Suppose that X has the property ACF. Let $\mathcal{C}$ be a non-empty compact subset of $\mathcal{K}(X)$. Then by [6, Theorem 0.2], $\cup\mathcal{C}$ is compact in X. By $ACF, \cup\mathcal{C}$ is finite; consequently $\mathcal{C}$ is finite. This implies that $\mathcal{K}(X)$ has ACF. Under $ACF, \mathcal{K}(X)$ coincides with the set of non-empty finite subsets of X. Therefore if X is a σ-closed discrete space, then so is $\mathcal{K}(X)$. Using Theorem 1, the case of (2) is similar to (1).

For the case of σ-CF^* p.b.s., we do not know whether the following (C) is true or not: (C) A space X has a σ-CF^* p.b. if and only if so does $\mathcal{K}(X)$. But we can show the equivalence of (A), (B) and (C).

Proposition. *Conjectures $(A), (B)$ and (C) are equivalent.*

Proof. (B) $\to$ (A) is due to Corollary 2. (A) $\to$ (C) is shown in the proof of Theorem 5. (C) $\to$ (B): Let $X = K \cup D$ be the same as in (B). Assume that X has a σ-CF^* p.b.. By (C), $\mathcal{K}(X)$ has a σ-CF^* p.b.. For each pair x, y of distinct points of D, let

$$\mathcal{C} = \{\{x,p\} : p \in K\} \quad \text{and} \quad \mathcal{C}' = \{\{y,p\} : p \in K\}.$$

Then $\mathcal{C}$ and $\mathcal{C}'$ are disjoint, infinite compact subsets of $\mathcal{K}(X)$. Then by the argument of the case (a) in the proof of Theorem 1, $\mathcal{K}(X)$ is an $\aleph_0$-space. Consequently X is an $\aleph_0$-space. But D is an uncountable discrete subset of X. So this is a contradiction. Hence we establish the equivalence of (A), (B) and (C).

With respect to the conjecture (C), if we weaken the condition CF^* families to CF families, then we have the following positive result:

Theorem 6. *A space X has a σ-CF p.b. if and only if so does $\mathcal{K}(X)$.*

Proof. If part is trivial. Only if part: Let $\mathcal{P} = \cup\{\mathcal{P}(n) : n \in N\}$ be a p.b. for X, where for each n $\mathcal{P}(n) \subset \mathcal{P}(n+1)$ and $\mathcal{P}(n)$ is CF in X. For each n let

$$\langle\mathcal{P}(n)\rangle = \{\langle P_1, \cdots, P_k\rangle : P_1, \cdots, P_k \in \mathcal{P}(n) \text{ and } k \in N\},$$

where

$$\langle P_1, \cdots, P_k\rangle = \{K \in \mathcal{K}(X) : K \subset \cup\{P_i : 1 \le i \le k\} \text{ and } K \cap P_i \ne \phi \text{ for each } i\}.$$

First we show that for each $n \langle\mathcal{P}(n)\rangle$ is CF in $\mathcal{K}(X)$. Let $\mathcal{C}$ be a non-empty compact subset of $\mathcal{K}(X)$. Then by [6, Theorem 0.2], $C = \cup\mathcal{C}$ is compact in X. Since $\mathcal{P}(n)$ is CF in X, we have

$$\mathcal{P}(n)|C = \{Q(1), \cdots, Q(k)\}.$$

Let $\{\mathcal{Q}(\delta(i)) : i = 1, \cdots, t\}$ be the totality of finite subfamilies of $\{Q(1), \cdots, Q(k)\}$ such that for each i

$$\langle\mathcal{Q}(\delta(i))\rangle = \{K \in \mathcal{K}(X) : K \subset \cup\mathcal{Q}(\delta(i)) \text{ and } K \cap Q \ne \phi \text{ for each } Q \in \mathcal{Q}(\delta(i))\}$$

intersects $\mathcal{C}$. Let $\langle P_1, \cdots, P_m \rangle \cap \mathcal{C} \neq \phi$, where $\{P_1, \cdots, P_m\} \subset \mathcal{P}(n)$. Then all P_i intersect C. So, there exists $\delta(i)$ with $1 \leq i \leq t$ such that

$$\{P_1, \cdots, P_m\}|C = \mathcal{Q}(\delta(i)).$$

Then we have

$$\langle P_1, \cdots, P_m \rangle \cap \mathcal{C} = \langle \mathcal{Q}(\delta(i)) \rangle \cap \mathcal{C}.$$

Hence $\langle \mathcal{P}(n) \rangle$ is CF in $\mathcal{K}(X)$.

Let $\hat{\mathcal{P}}$ be the totality of finite unions of members of $\cup\{\langle \mathcal{P}(n) \rangle : n \in N\}$. Then $\hat{\mathcal{P}}$ is easily seen to be a σ-CF family in $\mathcal{K}(X)$. We show that $\hat{\mathcal{P}}$ is a p.b. in $\mathcal{K}(X)$. Let $\mathcal{K} \subset \hat{U}$, where $\mathcal{K}$ is a compact subset of $\mathcal{K}(X)$ and $\hat{U}$ is an open subset of $\mathcal{K}(X)$. Then we show that there exists $\hat{P} \in \hat{\mathcal{P}}$ such that $\mathcal{K} \subset \hat{P} \subset \hat{U}$. We consider the following two cases:

Case 1: Let $\hat{U}$ be a basic open subset of $\mathcal{K}(X)$, i.e.,

$$\hat{U} = \langle U_1, \cdots, U_k \rangle, U_1, \cdots, U_k \in \tau(X).$$

Since $\mathcal{K}$ is compact and $\mathcal{K}(X)$ is regular, there exists a finite open cover $\{\hat{V}(1), \cdots, \hat{V}(n)\}$ of $\mathcal{K}$ such that

$$\mathcal{K} \subset bigcup_i \hat{V}(i) \subset \bigcup_i Cl(\hat{V}(i)) \subset \hat{U},$$

where each $\hat{V}(i)$ is a basic open subset of $\mathcal{K}(X)$. Let $1 \leq i \leq n$ be fixed for a while and let

$$\hat{V}(i) = \langle V_1, \cdots, V_s \rangle, V_1, \cdots, V_s \in \tau(X).$$

For the latter use, we notice the following two facts:

$$\begin{aligned} Cl(\hat{V}(i)) &= \langle \bar{V}_1, \cdots, \bar{V}_s \rangle \\ &= \{K \in \mathcal{K}(X) : K \subset \bigcup_i \bar{V}_i \text{ and} \\ &\quad K \cap \bar{V}_i \neq \phi \text{ for each } i\} \text{ ([3, Lemma 2.3.2]).} \end{aligned} \tag{1}$$

$$Cl(\hat{V}(i)) \subset \hat{U} \rightleftarrows \bigcup_{j=1}^{s} \bar{V}_j \subset \bigcup_{i=1}^{k} U_i \tag{2}$$

and for each i with $1 \leq i \leq k$ there exists $j(i)$ with $1 \leq j(i) \leq s$ such that $\bar{V}_{j(i)} \subset U_i$ ([3, Lemma 2.3.1]).

Let

$$N_0 = \{j(i) : i = 1, \cdots, k\}, N_1 = \{1, \cdots, s\} \setminus N_0.$$

By virtue of [6, Theorem 0.2]

$$K(i) = \cup\{K : K \in \mathcal{K} \cap Cl(\hat{V}(i))\}$$

is compact in X and it is easily seen that

$$K(i) \in \langle \bar{V}_1, \cdots, \bar{V}_s \rangle \subset \langle U_1, \cdots, U_k \rangle.$$

For each $j \in N_0$, choose $P_j \in \mathcal{P}$ such that

$$K(i) \cap \bar{V}_j \subset P_j \subset \cap\{U_i : j(i) = j\}.$$

For each $j \in N_1$, choose $P_j \in \mathcal{P}$ such that

$$K(i) \cap \bar{V}_j \subset P_j \subset \bigcup_{i=1}^{k} U_i.$$

Easily we have

$$\mathcal{K} \cap Cl(\hat{V}(i)) \subset \langle P_1, \cdots, P_s \rangle \subset \hat{U}$$

and

$$\langle P_1, \cdots, P_s \rangle \in \langle \mathcal{P}(t(i)) \rangle$$

for some $t(i) \in N$. Running i from 1 to n, we have

$$\mathcal{K} \subset \hat{P} \subset \hat{U}$$

for some $\hat{P} \in \hat{\mathcal{P}}$.

Case 2: Let $\hat{U}$ be an open subset of $\mathcal{K}(X)$. From the definition of the base for the topology of $\mathcal{K}(X)$, there exists a finite cover $\{\hat{U}(i) : i \in i, \cdots, h\}$ of $\mathcal{K}$ consisting of basic open subsets of $\mathcal{K}(X)$ such that $\mathcal{K} \subset \bigcup_i \hat{U}(i) \subset \hat{U}$. Since $\mathcal{K}$ is compact, there exists a closed cover $\{\mathcal{K}(i) : i = 1, \cdots, h\}$ of $\mathcal{K}$ such that $\mathcal{K}(i) \subset \hat{U}(i)$ for each i. By virtue of the case 1, for each i we can find $\hat{P}(i) \in \hat{\mathcal{P}}$ such that $\mathcal{K}(i) \subset \hat{\mathcal{P}}(i) \subset \hat{U}(i)$.

Hence we have $\mathcal{K} \subset \bigcup_i \hat{\mathcal{P}}(i) \subset \hat{U}$ and $\hat{P} = \bigcup_i \hat{P}(i) \in \hat{\mathcal{P}}$, because $\hat{\mathcal{P}}$ is closed under finite unions. This completes the proof.

References

1. L. Foged, *A characterization of closed images of metric spaces*, Proc. Amer. Math. Soc., **95** (1985), 487–490.
2. S. Lin, *A study of pseudobases*, Q & A in Gen. Top., **6** (1988), 81–97.
3. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., **71** (1951), 152–182.
4. E. Michael, *$\aleph_0$-spaces*, J. Math. Mech., **15** (1966), 983–1002.
5. T. Mizokami, *On CF families and hyperspaces of compact subsets*, Top. Appl., **35** (1990), 75–92.
6. I. Ntantu, *Cardinal functions on hyperspaces and function spaces*, Top. Proc., **10** (1985), 357–375.
7. M. Sakai, *Remarks on spaces with special type of k-networks*, preprint.

Department of Mathematics, Joetsu University of Education, Joetsu, Niigata 943, Japan

Ningde Teacher's College, Ningde Fujian, P. R. China