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K-Spaces Property of Product Spaces

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Abstract Let $\mathcal{K}$ be a class of spaces which are eigher a pseudo-open s-image of a metric space or a k-space having a compact-countable closed k-network. Let $\mathcal{K}'$ be a class of spaces which are either a Fréchet space with a point-countable k-network or a point-G_δ k-space having a compact-countable k-network. In this paper, we obtain some sufficient and necessary conditions that the products of finitely or countably many spaces in the class $\mathcal{K}$ or $\mathcal{K}'$ are a k-space. The main results are that
Theorem A *If $X, Y \in \mathcal{K}$. Then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*
Theorem B *The following are equivalent:*
(a) *$BF(\omega_2)$ is false.*
(b) *For each $X, Y \in \mathcal{K}', X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

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1 Introduction

In this paper all spaces are regular and T_1. Suppose X is a topological space, and $\mathcal{P}$ is a collection of subsets of X. $\mathcal{P}$ is called a k-network for X if $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is a closed (compact) k-network if $\mathcal{P}$ is a k-network for X where each element is closed (compact) in X. A space X is a K_ω-space if X has a countable cover $\{K_n\}$ of compact subsets such that $F \subset X$ is closed in X if and only if $F \cap K_n$ is closed for each K_n. A pair (X, Y) of spaces X and Y has the Tanaka's condition if one of three properties of below holds:

(1) X and Y are first countable spaces.

(2) X or Y is a locally compact space.

(3) X and Y are local K_ω-spaces.

Michae[1] posed the following question: Find a sufficient and necessary condition that the product space $X \times Y$ is a k-space for k-spaces X and Y. One of the successful results for the class of generalized metric spaces is the Tanaka's condition as follows.

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Theorem 1.1[2] *If X and Y are k-spaces with a σ-locally finite k-network. Then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

How to improve on Theorem 1.1 is the main direction for the study of products of k-spaces. Since Theorem 1.1 does not hold in the class of k and M_1-spaces, a real generalization for Theorem1.1 is in the class of quotient s-images of metric spaces or closed images of metric spaces.

For the class of quotient s-images of metric spaces, Chen[3] tried to prove the following conjecture.

Conjecture 1.2 *If X and Y are k-spaces with a point-countable closed k-network, then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

But, Gruenhage[4] pointed out that Chen's proof is not true. About Conjecture 1.2, we introduce a class $\mathcal{K}$, which denotes the class of spaces which are either a pseudo-open s-image of a metric space or a k-space having a compact-countable closed k-network, here a collection $\mathcal{P}$ of subsets of a space X being point-countable (compact-countable) whenever $x \in X$ (K is compact in X). Then $\{P \in \mathcal{P} : x \in P\}$ $(\{P \in \mathcal{P} : K \cap P \neq \emptyset\})$ is countable. In Section 2, we discuss the k-space property of products of finitely or countably many spaces in the class $\mathcal{K}$, which generalizes Theorem1.1 and some related results, and is a partial answer to Conjecture 1.2.

For the class of closed images of metric spaces, Gruenhage[5] proved the following theorem.

Theorem 1.3 *The following are equivalent:*

(a) *$BF(\omega_2)$ is false.*

(b) *$S_\omega \times S_{\omega 1}$ is not a k-space.*

(c) *If X and Y are the closed images of metric spaces, then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

A space is called a Lašnev space if it is a closed image of a metric space. A Lašnev space is equivalent to a Fréchet space with a σ-HCP k-network. Dai and Liu[6] obtained a k-space property of product spaces for the class of k-spaces with a σ-HCP k-network, which is similar to Theorem 1.3.

The product of two CW-complexes is closely associated with the k-space property of product spaces because Tanaka[7] proved that supposing X and Y are CW-complexes, then $X \times Y$ is a CW-complex if and only if it is a k-space. About the product of CW-complexes, Tanaka and Zhou[8] proved

Theorem 1.4 *The following are equivalent:*

(a) *$BF(\omega_2)$ is false.*

(b) *$I_\omega \times I_{\omega 1}$ is not a CW-complex.*

(c) *If X and Y are CW-complexes, then $X \times Y$ is a CW-complex if and only if (X, Y) has the Tanaka's condition.*

(d) *If X and Y are the closed images of CW-complexes, then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

As is well known, every CW-complex is dominated by a cover of compact metric subsets. If X is a closed image of a CW-complex, then X is also dominated by a cover of compact metric subsets. For the dominated family of a space, Tanaka[9] further proved

Theorem 1.5 *The following are equivalent:*

(a) *$BF(\omega_2)$ is false.*

(b) *If X and Y are dominated by Lašnev spaces, then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

Concerning a series of theorems above, we introduce a class $\mathcal{K}'$, which denotes the class of spaces which are either a Fréchet space with a point-countable k-network or a point-G_8 k-space with a compact-countable k-network, here a point-G_8 space being a space where each point is a G_8 set in the space. In Section 3, we discuss the k-space property of products of finitely or countably many spaces in the class $\mathcal{K}'$, which is a common generalization of Theorems 1.3–1.5.

We recall two canonical quotient spaces S_a and S_2. For $a \geq \omega$, let S_a be the quotient space obtained from the topological sum of a convergent sequences by identifying all the limit points with a single point ∞. Let $S_2 = (N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ isolated. A local base at $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n) : m \geq m_0\}$, and U is a neighborhood at 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$.

2 On the Class $\mathcal{K}$

By [10], if a space X is a k-space with a point-countable closed k-network, then it is a quotient s-image of a metric space; if X is a quotient s-image of a metric space, then it is a k-space with a point-countable k-network; if X is a k-space with a point-countable k-network, then every countably compact closed subset of X is compact metrizable in X, thus X is a sequential space, hence X has a countable tightness.

Lemma 2.1 *Let $\mathcal{P}$ be a point-countable k-network for a k-space X which is closed under finite intersections. Putting $\mathcal{F} = \{P \in \mathcal{P} : \overline{P}$ is compact in $X\}$. Then $\mathcal{F}$ is a k-network for X if and only if every first countable closed subspace of X is locally compact.*

Proof Necessity. We can assume that X is first countable.For each $x \in X$, by Proposition 3.2 in [10], $x \in (\cup \mathcal{F}')^{\circ}$ for some finite $\mathcal{F}' \subset \mathcal{F}$. Hence X is locally compact.

Sufficiency. Let K be compact in X. By Miščenko's lemma, a collection of minimal covers of K consisting of a finite subcollection of $\mathcal{P}$ is at most countable, say $\{\mathcal{P}_n\}$. For each $n \in N$, let $\mathcal{A}_n = \bigwedge_{i \leq n} \mathcal{P}_i, A_n = \cup \mathcal{A}_n$. Then $\mathcal{A}_n \subset \mathcal{P}, K \subset A_n$ and $\{\overline{A}_n\}$ is a network of K in X. We assert that some $\overline{A}_n$ is compact. If not, then each $\overline{A}_n$ is not countably compact, thus $\overline{A}_n$ contains a countable discrete closed subset D_n. Put

$$H = K \cup \left(\bigcup_{n \in N} D_n \right).$$

Then H is a first countable closed subspace of X, but H is not locally compact, a contradiction. Hence $\overline{A}_n$ is compact for some $n \in N$. If $K \subset U$ with U open in X. There exists $m \geq n$ such that $K \subset A_m \subset \overline{A}_m \subset U$, *i.e.*, a finite $\mathcal{A}_m \subset \mathcal{F}$ such that $K \subset \cup \mathcal{A}_m \subset U$, thus $\mathcal{F}$ is a k-network for X.

Lemma 2.2 (Tanaka[11], Lemma 4) *Suppose $X \times Y$ is a k-space with $t(X) \leq \omega$. Then the following condition (C_1) or (C_2) holds:*

(C_1) If $\{A_n\} \downarrow x$ in X, then there exists a nonclosed subset $\{a_n\}$ of X with $a_n \in A_n$ for each $n \in N$.

(C_2) If $\{B_n\}$ is a k-sequence in Y, then some $\overline{B}_n$ is countably compact.

Lemma 2.3 *Suppose X is a quotient s-image of a metric space. If X has (C_2) of Lemma* 2.2*, then X is a quotient s-image of a locally separable metric space.*

Proof Suppose $f : M \to X$ is a quotient s-mapping where M is a metric space. Let $\mathcal{B}$ be a σ-locally finite base for M. For each $x \in X$, take $z \in f^{-1}(x)$. Let $\{B_n : n \in N\} \subset \mathcal{B}$ such that

$B_{n+1} \subset B_n$ and $\{B_n\}$ is a local base at z in M. Then $\{f(B_n)\}$ is a k-sequence in X, thus some $\overline{f(B_n)}$ is countably compact by (C_2), so $f(B_n)$ is a separable metrizable subspace of X. Hence X has the weak topology with respect to a point-countable cover $\{P \in f(\mathcal{B}) : P$ is a separable metrizable subspace of $X\}$, then X is a quotient s-image of a locally separable metric space.

Lemma 2.4 (Tanaka[12], Theorem 4.4) *Suppose X is a quotient s-image of a locally separable metric space. If X contains no closed copy of S_ω and S_2, then X has a point-countable base.*

Lemma 2.5 *Let $Y \in \mathcal{K}$. If $S_\omega \times Y$ is a k-space, then Y has a σ-locally finite compact k-network.*

Proof Since $S_\omega \times Y$ is a k-space, every first countable closed subspace of Y is locally compact by Lemma 2.2. If Y has a compact-countable closed k-network, Y has a star-countable compact k-network by Lemma 2.1, then Y has a σ-locally finite compact k-network by Theorem 2.4 in [13]. If Y is a pseudo-open s-image of a metric space, Y is a closed s-image of a locally compact metric space by Corollary 1.4 in [14], thus Y has a compact-countable closed k-network, hence Y has a σ-locally finite compact k-network.

Theorem 2.6 *If $X, Y \in \mathcal{K}$, then $X \times Y$ is a k-space if and only if (X, Y) has the Tanaka's condition.*

Proof If (X, Y) has the Tanaka's condition, then $X \times Y$ is a k-space[2]. Now, let $X \times Y$ be a k-space.

(1) If X and Y contain closed copies of S_ω or S_2, then $S_\omega \times X$ and $S_\omega \times Y$ are k-spaces because S_ω is a perfect image of S_2. By Lemma 2.5 and Theorem 1.1, (X, Y) has the Tanaka's condition.

(2) If X contains a closed copy of S_ω or S_2, and Y contains no closed copy of S_ω and S_2, then $S_\omega \times Y$ is a k-space, thus Y has a σ-locally finite compact k-network by Lemma 2.5, so Y is a quotient s-image of a locally compact metric space. By Lemma 2.4 and Lemma 2.2, Y is a locally compact space, thus (X, Y) has the Tanaka's condition.

(3) If X and Y contain no closed copies of S_ω and S_2, by Lemma 2.2, there exist four cases as follows:

Case 1 X and Y satisfy (C_1). Then X and Y have point-countable base by (C_1) and Theorem 9.8 in [15].

Case 2 X satisfies (C_1) and (C_2). Then X has a point-countable base by (C_1), and X is locally compact by (C_2).

Case 3 Y satisfies (C_1) and (C_2). Then Y is locally compact.

Case 4 X and Y satisfy (C_2). Then X and Y are quotient s-images of metric spaces, thus X and Y have point-countable bases by Lemma 2.3 and Lemma 2.4.

In a word, (X, Y) has the Tanaka's condition.

In the second part of this section we discuss the k-space property of products of countably many spaces in the class $\mathcal{K}$. A sequence $\{X_i\}$ of spaces is called satisfying the Tanaka's condition, if $\{X_i\}$ has one of the following three properties:

(1) All X_i are first countable spaces.

(2) There is an $n \in N$ such that all X_i are compact $(i > n)$ and all X_i but at most an $i \leq n$ must be locally compact.

(3) There is an $n \in N$ such that all X_i are compact $(i > n)$ and all X_i are $K_\omega (i \leq n)$ locally.

If $\{X_i\}$ is a sequence of k-spaces having the Tanaka's condition, then $\prod_{i \in N} X_i$ is a k-space.

Lemma 2.7 *Let $\{X_i\}$ be a sequence of quotient s-images of metric spaces. If $\prod_{i\in N} X_i$ is a k-space, then $\prod_{i\geq n} X_i$ is either a compact space or a first countable space for some $n \in N$.*

Proof By Theorem 1.3 in [16], $(S_\omega)^\omega$ and $(S_2)^\omega$ are not a k-space, then X_i contains a closed copy of S_ω or S_2 for only finitely many $i \in N$, thus there exists $j \in N$ such that X_i contains no closed copy of S_ω and S_2 for each $i \geq j$. By a proof similar to that of the Theorem 2.6, either all X_i are first countable for each $i \geq j$, or $\prod_{i\geq m} X_i$ is locally compact for some $m \geq j$. Finally, $\prod_{i\geq n} X_i$ is compact for some $n \geq m$.

Corollary 2.8 *If X is a quotient s-image of a metric space, then X^ω is a k-space if and only if X is a first countable space.*

By Lemma 2.7 and Theorem 2.6, we have a countable product theorem on k-spaces.

Theorem 2.9 *If $\{X_i : i \in N\} \subset \mathcal{K}$, then $\prod_{i\in N} X_i$ is a k-space if and only if $\{X_i\}$ has the Tanaka's condition.*

Now, we discuss the gf-countability of products of countably many spaces in the class $\mathcal{K}$. Let X be a space. A collection $\mathcal{B}$ of subsets of X is said to be a weak base for X, if $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ satisfies that

(1) $x \in \cap\mathcal{B}_x$ for each $x \in X$.

(2) For each $U, V \in \mathcal{B}_x, W \subset U \cap V$ for some $W \in \mathcal{B}_x$.

(3) A subset G of X is open in X if and only if for each $z \in G$ there exists $B \in \mathcal{B}_z$ with $B \subset G$.

Here $\mathcal{B}_x$ is said to be a local weak base at x in X. If each $\mathcal{B}_x$ is countable, then X is said to be a gf-countable space. Every first countable space is gf-countable, and every Fréchet gf-countable space is first countable. Hence regarding the gf-countability of products of spaces in the class $\mathcal{K}$ we need only to discuss the gf-countability of products of countably many spaces which are gf-countable spaces with a compact-countable closed k-network. Every gf-countable space is a sequential space. A subset P of a space X is a sequential neighborhood at x in X, if $\{x_n\}$ is a sequence in X with $x_n \to x$, then $\{x\} \cup \{x_n : n \geq i\} \subset P$ for some $i \in N$. If $\mathcal{B}_x$ is a local weak base at x in X, then B is a sequential neighborhood at x in X for each $B \in \mathcal{B}_x$.

Lemma 2.10 *Let $\{X_i\}$ be a sequence of gf-countable spaces. If $\prod_{i\in N} X_i$ is a sequential space, then it is a gf-countable space.*

Proof Let $Z = \prod_{i\in N} X_i$. For each $i \in N$, let $\cup\{\mathcal{B}_{ix_i} : x_i \in X_i\}$ be a weak base for X_i, where each $\mathcal{B}_{ix_i}$ is countable. For each $z = (x_i) \in Z$, put

$$\mathcal{P}_z = \left\{ \left(\prod_{i\leq n} B_{ix_i} \right) \times \prod_{i>n} X_i : B_{ix_i} \in \mathcal{B}_{ix_i}, i \leq n \quad \text{and} \quad n \in N \right\}.$$

Then $\mathcal{P}_z$ is countable, and each element of $\mathcal{P}_z$ is a sequential neighborhood at z in Z. We shall show that each $\mathcal{P}_z$ is a local weak base at z in Z. If not, there exists a nonopen subset G of Z such that for each $y \in G$, there is a $W \in \mathcal{P}_y$ with $W \subset G$. Since Z is sequential, there is a sequence $\{z_n\}$ of $Z\backslash G$ with $z_n \to z \in G$, then $W \subset G$ for some $W \in \mathcal{P}_z$, and $z_n \in Z\backslash W$, hence W is not a sequential neighborhood at z in Z, a contradiction. Therefore Z is a gf-countable space.

By the proof of Lemma 2.10, we know that supposing $\{X_i\}$ is a sequence of spaces with a point-countable (compact-countable) weak base, if $\prod_{i\in N} X_i$ is a k-space, then $\prod_{i\in N} X_i$ has a point-countable (compact-countable) weak base.

Theorem 2.11 *If $\{X_i\}$ is a sequence of gf-countable spaces with a compact-countable closed k-network, then the following conditions are equivalent:*

(1) $\prod_{i\in N} X_i$ *is a gf-countable space.*

(2) $\prod_{i\in N} X_i$ *is a k-space.*

(3) $\{X_i\}$ *has the Tanaka's condition.*

If $\{X_i\}$ is a sequence of spaces having a compact-countable closed weak base, then the above-mentioned conditions are equivalent to

(4) $\prod_{i\in N} X_i$ *has a compact-countable closed weak base.*

Proof By Theorem 2.9 and Lemma 2.10 it is easy.

3 On The Class $\mathcal{K}'$

Let ${}^{\omega}\omega$ be the set of all functions from ω into ω. For two functions f and $g \in {}^{\omega}\omega$, we define $f \leq g$ if and only if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. $BF(\omega_2)$ means the following assertation.

$BF(\omega_2)$: If $F \subset {}^{\omega}\omega$ has cardinality less that ω_2, then there exists $g \in {}^{\omega}\omega$ such that $f \leq g$ for all $f \in F$.

It is known that CH implies that $BF(\omega_2)$ is false.

Lemma 3.1 *Suppose $S_\omega \times X$ is a k-space. Then the following are equivalent:*

(a) $BF(\omega_2)$ *is false.*

(b) *If X has a point-countable k-network $\mathcal{P}$, then there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $x \in (\cup\mathcal{P}_x)^0$ for each $x \in X$.*

(c) *If X is a Fréchet space with a point-countable k-network, then X is a local K_ω-space.*

(d) *If X has a compact-countable k-network, then X is a local K_ω-space.*

Proof Since $S_\omega \times X$ is a k-space, by Lemma 2.2, every first countable closed subspace of X is locally compact.

(a) $\rightarrow$ (b). Suppose $BF(\omega_2)$ is false. Then there exists a subcollection $\{f_a \in {}^{\omega}\omega : a < \omega_1\}$ of ${}^{\omega}\omega$ such that if $g \in {}^{\omega}\omega$, then there exists $a < \omega_1$ with $f_a(n) > g(n)$ for infinitely many $n \in \omega$. Suppose there exists $x \in X$ such that $x \notin (\cup\mathcal{P}_x)^0$ for every countable $\mathcal{P}_x \subset \mathcal{P}$. Since X has countable tightness, let N_0 be a Moore-Smith net converging to $x, x \notin N_0$ and N_0 be countable. Let $\mathcal{P}_0 = \{P \in \mathcal{P} : P \cap N_0 \neq \emptyset\}$. Then $\mathcal{P}_0$ is countable. By transfinite induction, we can choose a collection $\{N_a : a < \omega_1\}$ of subsets of X such that

(1) Moore-Smith net N_a converges to $x, x \notin N_a$ and N_a is countable for each $a < \omega_1$.

(2) P meets at most one N_a for each $P \in \mathcal{P}$.

Put $H_a = \{(m_n, i_a) : m \leq f_a(n) \quad \text{and} \quad i \leq n\}$. where m_n and i_a denote the mth term of the nth sequence in S_ω and the ith term of the N_a, respectively. Let $H = \bigcup_{a<\omega_1} H_a$. Then $H \subset S_\omega \times X$. If K is compact in $S_\omega \times X$, then there exists $n \in \omega$ and a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset L_n \times (\cup\mathcal{F})$, where $L_n = \{\infty\} \cup \{m_i : i \leq n \text{ and } m \in \omega\}$. By (2), $(L_n \times (\cup\mathcal{F})) \cap H$ is finite, and $K \cap H$ is finite, hence K is k-closed in $S_\omega \times X$. Now, suppose U is any open set in $S_\omega \times X$ containing (∞, x), then there exists a neighborhood U_g at ∞ in S_ω and a neighborhood U_x at x in X such that $U_g \times U_x \subset U$ where $U_g = \{\infty\} \cup \{m_n : m \geq g(n)\}$ for some $g \in {}^{\omega}\omega$, thus there exists $a < \omega_1$ with $f_a(n) > g(n)$ for infinitely many $n \in \omega$. Take $n_a \in U_x \cap N_a$, and choose $n' > n$ with $f_a(n') > g(n')$, then $(f_a(n')_{n'}, n_a) \in (U_g \times U_x) \cap H \subset U \cap H$, thus $(\infty, x) \in \overline{H} \backslash H$, and H is not closed in X, hence $S_\omega \times X$ is not a k-space, a contradiction.

(b) $\rightarrow$ (c). Suppose X is a Fréchet space with a point-countable k-network $\mathcal{P}$. By Lemma 2.1, we can assume that $\overline{P}$ is compact for each $P \in \mathcal{P}$. By (b), there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ with $x \in (\cup\mathcal{P}_x)^0$ for each $x \in X$. Let V be open in X such that $x \in V \subset \overline{V} \subset (\cup\mathcal{P}_x)^0$. Then

$\overline{V}$ is a σ-compact, Fréchet space with a point-countable k-network, hence $\overline{V}$ has a countable k-network by Theorem 5.2 in [10]. By Lemma 2.1, $\overline{V}$ has a countable compact k-network, and $\overline{V}$ is a K_ω-space, hence X is a local K_ω-space.

(b) $\to$ (d). Suppose X is a space with a compact-countable k-network $\mathcal{P}$. By Lemma 2.1, we can assume that $\overline{P}$ is compact for each $P \in \mathcal{P}$. By (b), $\mathcal{P}$ is locally countable, and X has a locally countable compact k-network, thus X is a local K_ω-space.

(c) or (d) $\to$ (a). Since $S_{\omega 1}$ is a non local K_ω, Fréchet space with a compact-countable k-network, if (c) or (d) holds, then $S_\omega \times S_{\omega 1}$ is not a k-space. By Theorem 1.3, $BF(\omega_2)$ is false.

Theorem 3.2 *The following are equivalent:*

(a) $BF(\omega_2)$ *is false.*

(b) *For each* $X, Y \in \mathcal{K}'$, $X \times Y$ *is a* k*-space if and only if* (X, Y) *has the Tanaka's condition.*

Proof Suppose $BF(\omega_2)$ is false, and $X, Y \in \mathcal{K}'$. If (X, Y) has the Tanaka's condition, then $X \times Y$ is a k-space. Now, let $X \times Y$ be a k-space.

(1) If X and Y contain closed copies of S_ω or S_2, then $S_\omega \times X$ and $S_\omega \times Y$ are k-spaces. By Lemma 3.1, X and Y are local K_ω-spaces.

(2) If X contains a closed copy of S_ω or S_2, and Y contains no closed copy of S_ω and S_2, then $S_\omega \times Y$ is a k-space and Y is a stronly Fréchet space by Theorems 3.1 and 1.5 in [12], so Y is a first countable space by Corollary 3.6 in [10], thus Y is a locally compact space by Lemma 2.2.

(3) If X and Y contain no closed copies of S_ω and S_2, then X and Y are first countable by the proof in (2).

In a word, (X, Y) has the Tanaka's condition.

Now, we assume that $BF(\omega_2)$ holds, then $S_\omega \times S_{\omega 1}$ is a k-space by Theorem 1.3. It is obvious that S_ω and $S_{\omega 1} \in \mathcal{K}'$, but $(S_\omega, S_{\omega 1})$ has not the Tanaka's condition.

Now, we discuss some applications of Theorem 3.2. Let X be a space, and let $\mathcal{O}$ be a cover of X. X is determined by $\mathcal{O}$, if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for each $C \in \mathcal{O}$. X is dominated by $\mathcal{O}$, if the union of any subcollection $\mathcal{O}'$ of $\mathcal{O}$ is closed in X, and the union is determined by $\mathcal{O}'$. Let $\mathcal{A} = \{A_a : a \in A\}$ be a collection of subsets of a space X. Then $\mathcal{A}$ is hereditarily closure-preserving if $\overline{\cup\{B_a : a \in A\}} = \cup\{\overline{B_a} : a \in A\}$ whenever $B_a \subset A_a$ for each $a \in A$. Every space is dominated by its a hereditarily closure-preserving closed cover. We shall write HCP instead of hereditarily closure-preserving.

Every Lašnev space has a σ-HCP k-network. If a space X has a σ-HCP k-network $\mathcal{P}$, let $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$, where each $\mathcal{P}_n$ is HCP in X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in N$. Put

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\}, \quad \mathcal{F}_n = \{P \backslash D_n : P \in \mathcal{P}_n\} \cup \{\{x\} : x \in D_n\}.$$

It is easy to check that $\bigcup_{n \in N} \mathcal{F}_n$ is a compact-countable k-network for X. If a space X is dominated by a cover $\{X_a : a < \beta\}$, for each $a < \beta$, let

$$Y_0 = X_0 \quad \text{and} \quad Y_a = X_a \backslash \cup \{X_\lambda : \lambda < a\}, \quad a > 0.$$

Then $\{Y_a : a < \beta\}$ is a compact-finite cover of X. If each X_a has a compact-countable k-network, then X has also a compact-countable k-network.

Corollary 3.3 *The following are equivalent:*

(a) $BF(\omega_2)$ *is false.*

(b) *If* X *and* Y *are dominated by* k*-spaces with a* σ*-HCP* k*-network, then* $X \times Y$ *is a* k*-space if and only if* (X, Y) *has the Tanaka's condition.*

Proof We prove only that if $BF(\omega_2)$ is false, X and Y are dominated by k-spaces with a σ-HCP k-network, and $X \times Y$ is a k-space, then (X, Y) has the Tanaka's condition. Since X and Y are dominated by spaces with a compact-countable k-network, X and Y have compact-countable k-networks. Since X and Y are dominated by σ-spaces, X and Y are also σ-spaces[17], hence X and Y are point-G_8 spaces. By Theorem 3.2, (X, Y) has the Tanaka's condition.

For the k-space property and the gf-countability of products of countably many spaces in the class $\mathcal{K}'$, by the proof of Theorem 2.9, Theorem 2.11 and Theorem 3.2, we have that

Theorem 3.4 *The following are equivalent:*

(a) $BF(\omega_2)$ *is false.*

(b) *If* $\{X_i : i \in N\} \subset \mathcal{K}'$, *then* $\prod_{i\in N} X_i$ *is a* k*-space if and only if* $\{X_i\}$ *has the Tanaka's condition.*

Theorem 3.5 *The following are equivalent:*

(a) $BF(\omega_2)$ *is false.*

(b) *If* $\{X_i\}$ *is a sequence of point-*G_8, gf*-countable spaces with a compact-countable* k*-network. Then* $\{X_i\}$ *has the Tanaka's condition if and only if one of two properties holds:*

(1) $\prod_{i\in N} X_i$ *is a* gf*-countable space.*

(2) $\prod_{i\in N} X_i$ *is a* k*-space.*

(c) *If* $\{X_i\}$ *is a sequence of point-*G_8 *spaces having a compact-countable weak base, then* $\{X_i\}$ *has the Tanaka's condition if and only if* $\prod_{i\in N} X_i$ *has a compact-countable weak base.*

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