ON THE QUOTIENT IMAGES OF NORMAL METRIC SPACES*

SHOU LIN

Received July, 1994

ABSTRACT. In this paper the main result is that every quotient map from a normal metric space is bi-quotient.

1. Introduction. In this paper, all spaces are assumed to be T_2, all maps continuous and onto. N denotes the set of natural numbers. A metric d on a metric space X is called normal if $d(A, B) > 0$ for every pair A, B of disjoint closed subsets of X. Aspace X is a normal metric space if it has a normal metric [2]. Mrowka obtained an equivalent condition for normal metric spaces as follows in [4].

Theorem 1.1. *Suppose X is a metric space. Then X is a normal metric space if and only if the set of all limit points of X is a compact set of X.*

Normal metric spaces have many important mapping theorems. First, recall some definitions. Let $f\colon X \to Y$ be a map. f is quotient if whenever $f^{-1}(F)$ is closed in X with $F \subset Y$, then F is closed in Y. f is pseudo-open if whenever $f^{-1}(y) \subset U$ with $y \in Y$ and U open in X, then $f(U)$ is a neighborhood of y in Y. f is (countably) bi-quotient if whenever $\mathcal{U}$ is a (countable) collection of open sets of X, and covers $f^{-1}(y)$ for some $y \in Y$, then $f(\cup\mathcal{U}')$ is a neighborhood of y in Y for some finite subcollection $\mathcal{U}'$ of $\mathcal{U}$. f is open (closed) if whenever V is open (closed) in X, then $f(V)$ is open (closed) in Y. It is well known that [6]

$$\begin{array}{ccccc} \text{open map} & \Longrightarrow & \text{bi-quotient map} & \Longrightarrow & \text{countably bi-quotient map} \\ & & & & \Downarrow \\ & & \text{closed map} & \Longrightarrow & \text{pseudo-open map} \\ & & & & \Downarrow \\ & & & & \text{quotient map} \end{array}$$

and none of the implications can be reversed.

Theorem 1.2. [1, 3, 7] *The following conditions are equivalent for a space X:*

(1) *X is a normal metric space.*
(2) *Every closed image of X is metrizable.*
(3) *Every pseudo-open image of X is metrizable.*
(4) *Every quotient image of X is metrizable.*

The following theorem is an equivalent form of Theorem 1.2 by maps.

Theorem 1.3. [3] *Suppose X is a metric space, then every quotient image of X is metrizable if and only if every quotient map from X is pseudo-open.*

---

*Project supported by the Mathematics Tianyuan Foundation of NNSF of CHINA.
1991 *Mathematics Subject Classification.* 54C10, 54E35.
Key words and phrases. Normal metric space, quotient map, bi-quotient map.

By Theorem 1.2 and Theorem 1.3, the following questions are raised:

Question 1.4. Is every quotient map from X (countably) bi-quotient if X is a normal metric space?

Question 1.5. Is every quotient map from X closed or open if X is a normal metric space?

Question 1.6. Is X a normal metric space if every (countably) bi-quotient image of X is metrizable?

The purpose of this paper is to give an affirmative answer to question 1.4, and negative answers to questions 1.5 and 1.6.

2. The Main Theorems.

Theorem 2.1. *Every quotient map from a normal metric space is bi-quotient.*

Proof. Suppose $f\colon X \to Y$ is a quotient map, where X is a normal metric space. X' denotes the set of all limit points of X, then X' is compact by Theorem 1.1. For each $y \in Y$, let $\mathcal{U}$ be a collection of open sets of X, which covers $f^{-1}(y)$. We can assume that $y \in f(X')$, otherwise y is an isolated point in Y, then $f(U)$ is a neighborhood of y in Y for each $U \in \mathcal{U}$ with $f^{-1}(y) \cap U \neq \emptyset$. Since $f^{-1}(U) \cap X'$ is compact, there is a finite subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $f^{-1}(y) \cap X' \subset \cup\mathcal{U}'$. Let

$$H = (Y \setminus f(X' \setminus \cup\mathcal{U}')) \cap f(\cup\mathcal{U}'),$$

then H is open in Y by the proof of Theorem 1 in [1], and $y \in H \subset f(\cup\mathcal{U}')$, hence $f(\cup\mathcal{U}')$ is a neighborhood of y in Y. Therefore f is bi-quotient. □

Remark. By Theorem 1.2 and [5, Theorem 4.3], we immediately know that every quotient map from a normal metric space is countably bi-quotient. But, a quotient map from a (locally compact) metric space onto a (compact) metric space can not be bi-quotient. For example, let Y be the closed unit interval $[0,1]$, and X be the topological sum of all convergent sequences in Y, and let f be the obvious map from X onto Y. Then f is a quotient (indeed, countably bi-quotient) map. Since any non-empty open subset of Y is uncountable, f is not bi-quotient.

Example 2.2. There exist a normal metric space X and a quotient map f from X such that f is neither closed nor open.

Construction. Take $X = I \times \omega$, where I is the unit interval, and ω is the set of finite orders. Let $\mathcal{B}$ be a countable base for I with usual Euclidean topology. For each $B \in \mathcal{B}$, $m \in N$, put

$$V(B,m) = B \times (\{0\} \cup \{n \in N : n \geq m\}),$$

and let

$$\mathcal{P} = \{\{x\} : x \in I \times N\} \cup \{V(B,m) \colon B \in \mathcal{B},\, m \in N\},$$

and the set X is endowed the topology generated by the base $\mathcal{P}$. Then X is a regular space, and $\mathcal{P}$ is a σ-discrete base for X. By the classic Bing Metrization Theorem, X is a metric space. Since the set of all limit points of X is I with usual Euclidean topology, which is compact in X, X is normal metric space by Theorem 1.1. Let $f\colon X \to I$ be a projective map, then f is quotient, but f is neither closed nor open.

Example 2.3. There is a metric space X such that

(1) X is not a normal metric space.
(2) Every countably bi-quotient image of X is metrizable.

Construction. Let $I_n = I$ with usual Euclidean topology for each $n \in N$. Take $X = \bigoplus_{n \in N} I_n$, then X is a metric space, and it is not a normal metric space by Theorem 1.1.

Suppose $f : X \to Y$ is a countably bi-quotient map, then Y is locally compact, hence regular, and also Y is locally metric space by the definition of the countable bi-quotient maps. While, a regular space Y is Lindelöf, hence paracompact. Then Y is a locally metric, paracompact space. Thus, as is well-known, Y is metrizable.

Acknowledgement. The author would like to thank the referee for his useful suggestions to improve the paper.

References

1. K. Alster, *Metric spaces all of whose decomposition are metric*, Bull. Acad. Pol. Sci. Math., **20**(1972), 395–400.
2. Y. Hattori and J. Nagata, *Special metrics*, In Recent Progress in General Topology, M. Husek, J. van MIll, Editors, North-Holland, 1992, pp. 353–368.
3. A. J. Jayanthan and V. Kannan, *Spaces every quotient of which is metrizable*, Proc. AMS, **103**(1988), 294–298.
4. S. G. Mrowka, *Normal metrics*, Amer. Math. Monthly, **72**(1965), 998–1001.
5. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology Appl., **1**(1971), 143–154.
6. F. Siwiec and V. J. Mancuso, *Relations among certain mappings and conditions for their equivalences*, General Topology Appl., **1**(1971), 23–41.
7. S. Willard, *Metric spaces all of whose decomposition are metric*, Proc. AMS, **21**(1969), 126–128.

Department of Mathematics, Ningde Teachers' College, Ningde, Fujian 352100 P. R. China