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Perfect Preimages of Some Spaces *)

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Abstract In this paper we study perfect preimages of some generalized metric spaces. The purpose is to give some conditions, under which the k-semistratifiable spaces and the spaces with a σ-closure-preserving k-network are preserved by perfect pre-mappings.

Key Words and Phrases Perfect Mapping, Submesocompact Space, Lašnev Space, G_δ-diagonal, k-semistratifiable Space, k-network

Mappings are powerful tools in studying generalized metric spaces. The perfect mapping, which is one of the most important mappings, plays among all continuous mappings, a role similar to that of compact spaces among all topological spaces. The purpose of this paper is to study some properties of the perfect preimages of some generalized metric spaces. In Section 2, we discuss the problem whether perfect preimages of spaces with a σ-closure-preserving k-network are still this kind of spaces under certain condition. In section 3, we discuss the relation between perfect preimages of Lašnev spaces and closed images of paracompact p-spaces. Some interesting questions in this field are posed.

By a space we shall always mean a regular and T_1 topological space. Mappings are continuous and onto. N denotes the set of all natural numbers.

§ 1. On Submesocompact Spaces

In this section we shall study a new covering property which will be used in the sequel.

For a topological space X, let $\mathscr{K}(X)=\{K\subset X: K$ is an non-empty compact subspace$\}$. A sequence $\{\mathscr{U}_n\}$ of covers of X is called a θ-sequence with respect to $\mathscr{K}(X)$, if there is an $n\in N$ such that $(\mathscr{U}_n)_K=\{U\in\mathscr{U}_n: U\cap K\neq\phi\}$ is finite for each $K\in\mathscr{K}(X)$.

Definition 1.1 A space X is a submesocompact space if every open cover of X has an open refining θ-sequence with respect to $\mathscr{K}(X)$.

It is obvious that

$$\text{mesocompact}\rightarrow\text{submesocompact}\rightarrow\text{submetacompact}.$$

Using a technique invented by Junnila in [1] for submetacompact spaces, we can obtain the

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following characterizations of submesocompact spaces.

Theorem 1. 1 The following conditions are equivalent for a space X:

(1) X is a submesocompact space;

(2) Every well-ordered open cover of X has an open refining θ-sequence with respect to $\mathscr{K}(X)$;

(3) Every directed open cover of X has a σ-closure-preserving closed refinement which is refined by $\mathscr{K}(X)$;

(4) Every open cover $\mathscr{U}$ of X has a σ-closure-preserving closed family $\mathscr{R}$ such that for each $K\in\mathscr{K}(X)$, there is a $P\in\mathscr{R}$ and a finite subfamily $\mathscr{U}'$ of $(\mathscr{U})_K$ with $K\subset P\subset\bigcup\mathscr{U}'$.

Proof It is proved that (1)↔(2)↔(3) in [2] and [3], and (4)→(3) is obvious. To complete the proof, it is sufficient to show (1)→(4). Let

$$\mathscr{P}=\{\mathscr{U}'\subset\mathscr{U}:\ \mathscr{U}'\text{ is finite}\}.$$

$\mathscr{U}$ has an open refining θ-sequence $\{\mathscr{U}_i\}$ with respect to $\mathscr{K}(X)$. For each $i,j\in N$, put

$$C_{i,j}=\bigcup\{K\in\mathscr{K}(X):\ \text{ord}(K,\mathscr{U}_i)\leqslant j\},$$
$$\mathscr{F}_{i,j}=\{F_{i,j}(\mathscr{U}'):\ \mathscr{U}'\in\mathscr{P}\},$$

where $F_{i,j}(\mathscr{U}')=\{x\in C_{i,j}:\ \text{st}(x,\mathscr{U}_i)\subset\bigcup\mathscr{U}'\}$. Then $C_{i,j}$ is closed in X and $\mathscr{F}_{i,j}$ is a closure-preserving family of closed subsets of X. In fact, it is obvious that each $C_{i,j}$ is closed in X. For each $\mathscr{P}'\subset\mathscr{P}$, let

$$y\in\overline{\bigcup\{F_{i,j}(\mathscr{U}'):\ \mathscr{U}'\in\mathscr{P}'\}};$$

then $\text{ord}(y,\mathscr{U}_i)\leqslant j$, and thus $(\bigcap(\mathscr{U}_i)_y)\bigcap F_{i,j}(\mathscr{U}')\neq\phi$ for some $\mathscr{U}'\in\mathscr{P}'$. Take $z\in(\bigcap(\mathscr{U}_i)_y)\bigcap F_{i,j}(\mathscr{U}')$; then $(\mathscr{U}_i)_y\subset(\mathscr{U}_i)_z$ and $\text{st}(z,\mathscr{U}_i)\subset\bigcup\mathscr{U}'$. So $\text{st}(y,\mathscr{U}_i)\subset\bigcup\mathscr{U}'$, and hence $y\in F_{i,j}(\mathscr{U}')$. Therefore

$$\overline{\bigcup\{F_{i,j}(\mathscr{U}'):\ \mathscr{U}'\in\mathscr{P}'\}}=\bigcup\{F_{i,j}(\mathscr{U}'):\ \mathscr{U}'\in\mathscr{P}'\}.$$

Put $\mathscr{R}=\bigcup\{\mathscr{F}_{i,j}:\ i,j\in N\}$; then $\mathscr{R}$ is a σ-closure-preserving family of closed subsets of X. For each $K\in\mathscr{K}(X)$, there are $i,j\in N$ with $\text{ord}(K,\mathscr{U}_i)\leqslant j$. Thus, for some finite $\mathscr{U}'\subset(\mathscr{U})_K$, $K\subset\bigcup(\mathscr{U}_i)_K\subset\bigcup\mathscr{U}'$, and $K\subset F_{i,j}(\mathscr{U}')\subset\bigcup\mathscr{U}'$.

Theorem 1. 2 Suppose $f:X\to Y$ is a perfect mapping. Then X is a submesocompact space if and only if Y is a submesocompact space.

Proof Suppose X is a submesocompact space. For each directed open cover $\mathscr{U}$ of Y, $f^{-1}(\mathscr{U})$ is a directed open cover of X, and so $f^{-1}(\mathscr{U})$ has a σ-closure-preserving closed refinement $\mathscr{P}$ which is refined by $\mathscr{K}(X)$. Then $f(\mathscr{P})$ is a σ-closure-preserving closed refinement of $\mathscr{U}$ which is refined by $\mathscr{K}(Y)$, and Y is a submesocompct space.

Conversely, suppose Y is a submesocompact space. Let $\mathscr{U}$ be a well-ordered open cover of X. There is a $U_y\in\mathscr{U}$ and an open neighborhood V_y of y in Y with $f^{-1}(V_y)\subset U_y$ for each $y\in Y$. Put $\mathscr{V}=\{V_y:\ y\in Y\}$; then the open cover $\mathscr{V}$ of Y has an open refining θ-sequence $\{\mathscr{V}_i\}$ with respect to $\mathscr{K}(Y)$, and hence $\{f^{-1}(\mathscr{V}_i)\}$ is an open refining θ-sequence of $\mathscr{U}$ with respect to $\mathscr{K}(X)$, and X is a submesocompact space.

Definition 1. 2 A space X is called of K-G_δ^*-diagonal (G_δ^*-diagonal, G_δ-diagonal, resp.) if there is a sequence $\{\mathscr{U}_i\}$ of open covers of X with $K=\bigcap_{i\in N}\overline{\mathrm{st}(K,\mathscr{U}_i)}$ for each $K\in\mathscr{K}(X)$ ($\{x\}=\bigcap_{i\in N}\overline{\mathrm{st}(x,\mathscr{U}_i)}$, $\{x\}=\bigcap_{i\in N}\mathrm{st}(x,\mathscr{U}_i)$ for each $x\in X$, resp.). The $\{\mathscr{U}_i\}$ is called a K-G_δ^*-diagonal sequence (G_δ^*-diagonal sequence, G_δ-diagonal sequence, resp.) for X.

It is easy to check that

$$K\text{-}G_\delta^*\text{-diagonal}\rightarrow G_\delta^*\text{-diagonal}\rightarrow G_\delta\text{-diagonal}.$$

It is well known that a submetacompact space with a G_δ-diagonal has a G_δ^*-diagonal. Using the same argument as in [4], Theorem 2.11, we have the following result on K-G_δ^*-diagonals.

Proposition 1.1 A submesocompact space with a G_δ-diagonal has a K-G_δ^*-diagonal.

By Theorem 1.1, the following question is interesting.

Question 1.1 Is a space X submesocompact if every directed open cover of X has a σ-cushioned refinement which is refined by $\mathscr{K}(X)$?

§ 2. Perfect Preimages of Spaces with σ-closure-preserving k-networks

Spaces with σ-closure-preserving k-networks, as a common generalization of stratifiable spaces and $\aleph$-spaces, have many important properties. For example, they are preserved by closed mappings; a Fréchet space with a σ-closure-preserved k-network is stratifiable. But there is an open problem: Is a space with a G_δ-diagonal a space with a σ-closure-preserving k-network if it is a perfect preimage of a space with a σ-closure-preserving k-network? We give an affirmative answer to this question.

Definition 2.1 A collection $\mathscr{P}$ of subsets of a space X is a k-network for X, if for each $K\in\mathscr{K}(X)$ and each open neighborhood U of K in X, there is a finite subfamily $\mathscr{P}'$ of $\mathscr{P}$ such that $K\subset\bigcup\mathscr{P}'\subset U$.

Theorem 2.1 Suppose $f: X\rightarrow Y$ is a perfect mapping. If Y is a space with a σ-closure-preserving k-network, then X is a space with a σ-closure-preserving k-network if and only if X has a G_δ-diagonal.

Proof Necessity is obvious.

Sufficiency. Suppose X has a G_δ-diagonal. Since Y is a space with a σ-closure-preserving k-network, by Theorem 1.1, Y is a submesocompact space. From Theorem 1.2 and Proposition 1.1, X has a K-G_δ^*-diagonal. Let $\{\mathscr{U}_n\}$ be a K-G_δ^*-diagonal sequence for X with $\mathscr{U}_{n+1}<\mathscr{U}_n$ for each $n\in N$. For each $n\in N$, by Theorems 1.1 and 1.2, there is a σ-closure-preserving family $\mathscr{P}_n$ of closed subsets of X such that for each $K\in\mathscr{K}(X)$ there is $P\in\mathscr{P}_n$ with $K\subset P\subset\mathrm{st}(K,\mathscr{U}_n)$. Denote $\mathscr{P}_n$ by $\bigcup\{\mathscr{P}(n,m): m\in N\}$, where each $\mathscr{P}(n,m)$ is a closure-preserving family of closed subsets of X. Let $\mathscr{Q}$ be a σ-closure-preserving closed k-network for Y, and denote $\mathscr{Q}$ by $\bigcup\{\mathscr{Q}_k: k\in N\}$, where each $\mathscr{Q}_k$ is a closure-preserving family of closed subsets of Y and $\mathscr{Q}_k\subset\mathscr{Q}_{k+1}$. For each $k\in N$, put

$$\mathscr{G}_k=f^{-1}(\mathscr{Q}_k);$$

then $\mathscr{G}_k$ is a closure-preserving family of closed subsets of X. Put

$$\mathscr{H}=\bigcup\{\mathscr{H}(n,m,k):n,m,k\in N\},$$

where $\mathscr{H}(n,m,k)=\mathscr{P}(n,m)\wedge\mathscr{G}_k$.

It is easy to check that $\mathscr{H}$ is a σ-closure-preserving family of closed subsets of X. We shall prove that $\mathscr{H}$ is a k-network for X. For $K\subset V$ with $K\in\mathscr{K}(X)$ and V open in X, put $C=f^{-1}f(K)$. Since $K=C\cap(\bigcap_{n\in N}\overline{\mathrm{st}(K,\mathscr{U}_n)})$, one has $C\subset V\cup(\bigcup_{n\in N}(C\backslash\overline{\mathrm{st}(K,\mathscr{U}_n)}))$, and $C\subset V\cup(C\backslash\overline{\mathrm{st}(K,\mathscr{U}_{n_0})})$ for some $n_0\in N$, and thus $C\cap\overline{\mathrm{st}(K,\mathscr{U}_{n_0})}\subset C\cap V$. There is an open neighborhood U of $C\backslash V$ in X with $U\cap\overline{\mathrm{st}(K,\mathscr{U}_{n_0})}=\phi$ because $(C\backslash V)\cap\overline{\mathrm{st}(K,\mathscr{U}_{n_0})}=\phi$. Put $W=V\cup U$, then $C\subset W$, and $f(K)\subset Y\backslash f(X\backslash W)$, and hence there is a $k_0\in N$ and a finite subcollection $\mathscr{Q}'_{k_0}$ of $\mathscr{Q}_{k_0}$ with $f(K)\subset\bigcup\mathscr{Q}'_{k_0}\subset Y\backslash f(X\backslash W)$. Then $K\subset\bigcup\{f^{-1}(Q):Q\in\mathscr{Q}'_{k_0}\}\subset W$. On the other hand, by the construction of $\mathscr{P}_{n_0}$, there are $m_0\in N$ and $P\in\mathscr{P}(n_0,m_0)$ with $K\subset P\subset\mathrm{st}(K,\mathscr{U}_{n_0})$, and so

$$K\subset\bigcup\{P\cap f^{-1}(Q):Q\in\mathscr{Q}'_{k_0}\}\subset V.$$

Therefore $\mathscr{H}$ is a σ-closure-preserving closed k-network for X.

Definition 2. 2[5] A space X is k-semistratifiable if there is a function G which assigns, to each $n\in N$ and each closed subset H of X, an open subset $G(n,H)$ containing H such that

(1) $H=\bigcap_{n\in N}G(n,H)$;

(2) $H_1\subset H_2\Rightarrow G(n,H_1)\subset G(n,H_2)$ for each $n\in N$;

(3) $K\in\mathscr{K}(X)$ and $K\cap H=\phi\Rightarrow K\cap G(n,H)=\phi$ for some $n\in N$.

k-semistratifiable spaces can be characterized by the concept of the pair of k-networks.

Definition 2. 3 A collection $\mathscr{F}$ of ordered pairs (F_1,F_2) of subsets of a space X is a pair of k-networks for X, if for each $K\in\mathscr{K}(X)$ and each open neighborhood U of K in X, there is a finite subcollection $\{(F_{1,i},F_{2,i}):i\leqslant n\}$ of $\mathscr{F}$ such that

$$K\subset\bigcup_{i\leqslant n}F_{1,i}\subset\bigcup_{i\leqslant n}F_{2,i}\subset U.$$

$\mathscr{F}$ is cushioned if for each $\mathscr{F}'\subset\mathscr{F}$, $(\overline{\bigcup\{F_1:(F_1,F_2)\in\mathscr{F}'\}})\subset\bigcup\{F_2:(F_1,F_2)\in\mathscr{F}'\}$. A cushioned $\mathscr{F}$ is closed if F_2 is closed in X for each $(F_1,F_2)\in\mathscr{F}$.

Theorem 2. 2[6] A space is k-semistratifiable if and only if it has a σ-cushioned pair k-network.

Using the similar method to that in Theorem 2. 1, we have the following result on k-semistratifiable spaces.

Theorem 2. 3 Suppose $f:X\to Y$ is a perfect mapping. If Y is a submesocompact k-semistratifiable space, then X is a k-semistratifiable space if and only if X has a G_δ-diagonal.

Question 2. 1 (1) Is a k-semistratifiable space a submesocompact space?

(2) Is a k-semistratifiable space a space with a σ-closure-preserving k-network (see [6])?

Remark 2. 1[7] We have proved that an orthocompact k-semistratifiable space is a space with a σ-closure-preserving k-network.

§ 3. Perfect Preimages of Lašnev Spaces

In this section we discuss the commucativity on mappings. A Lašnev space is a closed image of a metric space. We have known that perfect preimages of metric spaces can be characterized by paracompact M-spaces(=paracompact p-spaces). The classes of Lašnev spaces and paracompact M-spaces, which are all determined by metric spaces under suitable mappings, induce the following conjecture: does the class of perfect preimages of Lašnev spaces coincide with the class of closed images of paracompact M-spaces? The purpose of this section is to give a negative answer to the above conjecture.

Lašnev spaces can be characterized by the concept of k-networks. To discuss the property of closed images of paracompact M-spaces, we introduce a concept of (mod$\mathscr{K}$) k-networks, which is similar to the concept of (mod$\mathscr{K}$) networks.

Definition 3. 1 A collection $\mathscr{P}$ of subsets of a space X is a (mod$\mathscr{K}$) k-network for X, if there is $\mathscr{K}\subset\mathscr{K}(X)$ such that $\mathscr{K}(X)$ is refined by $\mathscr{K}$ and if $K\subset U$ with $K\in\mathscr{K}$ and U open in X, then $K\subset\bigcup\mathscr{P}'\subset U$ for some finite $\mathscr{P}'\subset P$.

Proposition 3. 1 The closed images of paracompact M-spaces have a σ-hereditarily·closure-preserving (mod$\mathscr{K}$) k-network.

Proof Suppose $f: X\to Y$ is a closed mapping, where X is a paracompact M-space. Then there is a metric space M and a perfect mapping p from X onto M. Let $\mathscr{B}$ be a σ-locally finite base of M, and put

$$\mathscr{P}=f(p^{-1}(\mathscr{B})),\qquad \mathscr{K}=f(p^{-1}(\mathscr{K}(M)));$$

then $\mathscr{P}$ is a σ-hereditarily closure-preserving family of subsets of Y, and $\mathscr{K}\subset\mathscr{K}(Y)$. For each $K\in\mathscr{K}(Y)$, there is $L\in\mathscr{K}(X)$ with $f(L)=K$[8] because f is a closed mapping on paracompact space, and then $K\subset f(p^{-1}(p(L)))\in\mathscr{K}$. It is easy to check that if $K\subset U$ with $K\in\mathscr{K}$ and U open in Y, then $K\subset f(p^{-1}(\bigcup\mathscr{B}'))\subset U$ for some finite $\mathscr{B}'\subset\mathscr{B}$, and hence $\mathscr{P}$ is a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network for Y.

Theorem 3. 1 A space X has a σ-hereditarily closure-preserving k-network if and only if X is a space with a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network and a G_δ-diagonal.

Proof It is sufficient to show the sufficiency. Suppose X has a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network and X has a G_δ-diagonal. Let $\mathscr{G}$ be a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network with respect to $\mathscr{K}$ by compact subsets of X, which satisfies the condition of Definition 3. 1. By the regularity of X, we can assume that $\mathscr{G}$ is a collection of closed subsets of X. First of all, we show that each open cover $\mathscr{U}$ of X has a σ-hereditarily closure-preserving closed refinement $\mathscr{P}$ such that for each $K\in\mathscr{K}(X)$, there is a finite subfamily $\mathscr{P}'$ of $\mathscr{P}$ with $K\subset\bigcup\mathscr{P}'$. Take an open cover $\mathscr{V}$ of X with $\overline{\mathscr{V}}<\mathscr{U}$. Put

$$\mathscr{Q}=\{G\in\mathscr{G}: \text{there is a finite subfamily } \mathscr{V}' \text{ of } \mathscr{V} \text{ with } G\subset\bigcup\mathscr{V}'\}.$$

For each $G\in\mathscr{Q}$, take a finite subfamily $\mathscr{V}(G)$ of $\mathscr{V}$ with $G\subset\bigcup\mathscr{V}(G)$. Denote $\mathscr{V}(G)$ by

$\{V_i(G): i \leqslant k(G)\}$. Put $\mathscr{P} = \{\overline{G \cap V_i(G)}: G \in \mathscr{Q}, i \leqslant k(G)\}$. It is easy to check that $\mathscr{P}$ is a σ-hereditarily closure-preserving closed refinement of $\mathscr{U}$, and if $K \in \mathscr{K}(X)$ then $K \subset \bigcup \mathscr{P}'$ for some finite $\mathscr{P}' \subset \mathscr{P}$.

From the above proof we see that every directed open cover of X has a σ-closure-preserving closed refinement which is refined by $\mathscr{K}(X)$. By Theorem 1.1 and Proposition 1.1, X has a K-G_δ^*-diagonal. By the similar method to that used in Theorem 2.1, we can prove that X has a σ-hereditarily closure-preserving k-network.

Example 3.1 There is a space X which is a perfect preimage of a Lašnev space, but not a closed image of a paracompact M-space.

Let X be the space $S_{\omega_1} \times I$ described in [9], which does not have a σ-hereditarily closure-preserving k-network. Since the projection $p: S_{\omega_1} \times I \to S_{\omega_1}$ is a perfect mapping, X is a perfect preimage of Lašnev space S_{ω_1}. By Theorem 3.1, X does not have a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network because X has a G_δ-diagonal. Thus X is not a closed image of a paracompact M-space by Proposition 3.1.

Example 3.2 There is a space X which is a closed image of a paracompact M-space, but not a perfect preimage of a Lašnev space.

Let X be the space Y described in Example 4.18 of [4]. It is a closed image of a paracompact M-space, but not a Σ-space. Since a Lašnev space is a Σ-space and a perfect preimage of a Σ-space is still a Σ-space, X is not a perfect preimage of a Lašnev space.

Foged[10] proved that a space is a Lašnev space if and only if it is a Fréchet space with a σ-hereditarily closure-preserving k-network. The following question is raised.

Question 3.1 Can a singly *bi*-k-space with a σ-hereditarily closure-preserving (mod$\mathscr{K}$) k-network be characterized as a closed image of a paracompact M-space? Here a space is a singly *bi*-k-space if it is a pseudo-open image of a paracompact M-space.

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