南京大学学报数学半年刊
第12卷 第1期 JOURNAL OF NANJING UNIVERSITY Vol. 12, No. 1
1995年5月 MATHEMATICAL BIQUARTERLY May, 1995

CLOSED IMAGES OF LOCALLY COMPACT METRIC SPACES*

Lin Shou
(Ningde Normal College)

Abstract In this paper the main result is that a T_2 —space X has a σ—hereditarily closure—preserving compact k—network if and only if X has a σ—hereditarily closure—preserving closed k—network, and each closed metric subspace of X is locally compact. As its an application, we obtain a new characterization on closed images of locally compact metric spaces.

Keywords K—network, locally compact space, metric space, closed mapping.

A study on images of metric spaces is one of central questions on general topology. Every quotient image of a metric space is actually the quotient image of a locally compact metric space[3]. However, every closed image of a metric space need not be the closed image of a locally compact metric space. A nice relation betweem closed images of metric spaces and closed images of locally compact metric spaces was established by Y. Tanaka in[8] as follows.

Theorem A A T_2 space X is a closed image of a locally compact metric space if and only if X is a closed image of a metric space, and each closed metric subspace of X is locally compact.

A characterization of closed images of mertic spaces was obtained by L. Foged in [2] as follows.

Theorem B A T_2 space X is a closed image of a metric space if and only if X is a Fréchet space with a σ—HCP closed k—network.

The purpose of this paper is to establish the further characterization on closed images of locally compact metric spaces. We prove that a T_2 space X has a σ—HCP compact k—network if and only if X has a σ—HCP closed k—network, and each closed metric subspace of X is locally compact. First, recall some definitions.

---

* 收稿日期:1994—04—03;
Project supported by the Mathematics "Tian Yuan" Fund of NNSF of China and NSF of Fujian Province, China

In this paper, all spaces are T_2, and all mappings are continuous and onto. N denotes the set of all natural numbers.

Definition 1 Let $f: X \to Y$, f is closed, if $f(F)$ is closed in Y whenever F is closed in X; f is compact, if $f^{-1}(y)$ is compact in X whenever $y \in Y$; f is perfect, if f is closed and compact.

Definition 2 Suppose $\mathscr{P}$ is a family of subsets of a space X. $\mathscr{P}$ is a k-network for X, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$. $\mathscr{P}$ is a closed (compact) k-network for X, if $\mathscr{P}$ is a k-network consisting of closed (compact) subsets of X.

Defifnition 3 Suppose $\mathscr{P}$ is a family of subsets of a space X. $\mathscr{P}$ is discrete (locally finite), if whenever $x \in X$, there exists an open neighborhood U of x in X that intersects at most one (finite) element of $\mathscr{P}$. $\mathscr{P}$ is HCP (i. e., hereditarily closure-preserving) if whenever $H(P) \subset P \in \mathscr{P}$, then $\bigcup \{\overline{H(P)}: P \in \mathscr{P}\} = \overline{\bigcup \{H(P): P \in \mathscr{P}\}}$.

Clearly, for a space,

discrete family→locally finite family→HCP family.

A σ-discrete (locally finite, HCP) family is a family that is the union of countably many discrete (locally finite, HCP) families.

Theorem A space X has a σ-HCP compact k-network if and only if X has a σ-HCP closed k-network, and each closed metric subspace of X is locally compact.

Proof Necessity. Suppose a space X has a σ-HCP compact k-network. Obviously, X has a σ-HCP closed k-network. If A is a closed metric subspace of X, then A has a σ-HCP compact k-network, thus there exists a paracompact, locally compact space Z and a closed mapping f from Z onto A by Theorem 2 in [5]. From paracompactness of Z and first countability of A, we can assume that f is perfect[6], then A is locally compact. Hence each closed metric subspace of X is locally compact.

Sufficiency. Suppose a space X has a σ-HCP closed k-network, and each closed metric subspace of X is locally compact. Let $\mathscr{P} = \bigcup_{n \in N} \mathscr{P}_n$ be a closed k-network for X, where each $\mathscr{P}_n$ is HCP in X, and $X \in \mathscr{P}_n \subset \mathscr{P}_{n+1}$. For each $n \in N$, put

$D_n = \{x \in X: \mathscr{P}$ is not point-finite at $x\}$,

$$\mathscr{R} = \{P \backslash D_n: P \in \mathscr{P}_n, n \in N\} \cup \{\{x\}: x \in D_n, n \in N\},$$

then, from the proof of Theorem in [4], we have the following facts:

(1) D_n is σ-*discrete in* X.

(2) $K \cap D_n$ is finite if K is compact in X.

(3) For a finite $\mathscr{F} \subset \mathscr{R}$, there are $m \in N$, $P \in \mathscr{P}_m$ and $D \subset D_m$ such that $\bigcap \mathscr{F} = (P \backslash D_m) \cup D$.

Define

$$\mathscr{H} = \{R \in \mathscr{R}: \overline{R} \text{ is compact in X}\}.$$
$$\mathscr{K} = \{\overline{H}: H \in \mathscr{H}\}.$$

Then, by (1), $\mathscr{H}$ is a σ—HCP family of compact subsets of X, we shall prove that $\mathscr{H}$ is a k—network for X.

For $K \subset U$ with K compact and U open in X, since $\mathscr{R}$ is a point—countable cover of K, by a result of A. S. Miščenko in [7], there are only countably many minimal finite subfamilies of $\mathscr{R}$ covering K, say $\{\mathscr{R}_i : i \in N\}$. For each $n \in N$, let $A_n = \bigcup (\bigwedge_{i \leq n} \mathscr{R}_i)$, then $\{\overline{A_n}\}$ is a descending sequence of closed subsets of X. If V is open in X with $K \subset V$, then $K \bigcup \mathscr{P}' \subset V$ for some finite $\mathscr{P}' \subset \mathscr{P}_i$.

Thus

$$K \subset (\bigcup \{P \backslash D_i : P \in \mathscr{P}'\}) \bigcup (K \cap D_i)$$
$$\subset (\bigcup \{\overline{P \backslash D_i} : P \in \mathscr{P}'\}) \bigcup (K \cap D_i) \subset V.$$

By (2), there is $n \in N$ such that

$$\mathscr{R}_n \subset \{P \backslash D_i : P \in \mathscr{P}'\} \bigcup \{\{x\} : x \in K \cap D_i\},$$

so $K \subset \overline{A_n} \subset V$, and $\{\overline{A_n}\}$ is a network of K in X. We assert that $\overline{A_n}$ is countably compact for some $n \in N$. In fact, if not, then for each $n \in N$, $\overline{A_n}$ has a countable subset B_n which is a closed discrete subspace in X. Put

$$B = K \bigcup (\bigcup_{n \in N} B_n),$$

then B is a closed metric subspace of X[9], and B is not locally compact, a contradiction. Hence there exists $n \in N$ such that $\overline{A_n} \subset U$ and $\overline{A_n}$ is countably compact. Since X is subparacompact, $\overline{A_n}$ is compact. Since A_n is a finite union of finite intersections of elements of $\mathscr{R}$, by (3), there are a finite $\mathscr{R}' \subset \mathscr{R}$ and some $D \subset D_m$ such that $A_n = (\bigcup \mathscr{R}') \bigcup D$. Put

$$\mathscr{H}' = \mathscr{R}' \bigcup \{\{x\} : x \in K \cap D\},$$

then $\mathscr{H}'$ is a finite subfamily of $\mathscr{H}$ and $K \subset U\{\overline{H} : H \in \mathscr{H}'\} \subset U$. Therefore, $\mathscr{H}$ is a k—network for X, and X has a σ—HCP compact k—network.

Corollary 1 The following properties are equivalent for a space X,

(1) X has a σ—discrete compact k—network.

(2) X has a σ—locally finite compact k—network.

(3) X has a σ—discrete closed k—network, and each closed metric subspace of X is locally compact.

(4) X has a σ—locally finite closed k—network, and each closed metric subspace of X is locally compact.

Proof By the proof of the above Theorem, we have that (1)⇔(3) and (2)⇔(4). By Theorem 4 in [1], we have that (3) ⇔(4).

Corollary 2 The following properties are equivalent for a space X,

(1) X is a closed image of a locally compact metric space.

(2) X is a closed image of a metric space, and each closed metric subspace of X is locally compact.

(3) X is a Fréchet space with a σ—HCP compact k—network.

(4) X is a Fréchet space with a σ-HCP closed k-network, and each closed metric subspace of X is locally compact.

References

1 Foged, L., Characterizations of ℵ-spaces, Pacific J. Math., 110(1984), 59~63.

2 Foged, L., A characterization of closed images of metric spaces, Proc. Amer. Math. Soc., 95(1985), 487~490.

3 Franklin, S. P., Spaces in which sequences suffice, Fund. Math., 57(1965), 107~115.

4 Lin, S., A decomposition theorem for Σ^*-spaces, Topology Proceedings, 16(1991), 125~128.

5 Lin, S., On the images of paracompact locally compact spaces, J. Math. (PRC) (in chinese), 12 (1992), 281~286.

6 Michael, E., A note on closed maps and compact sets, Israel J. Math., 2(1964), 173~176.

7 Miščenko, A. S., Spaces with a point countable base, Soviet Math. Dokl., 3(1962), 855~858.

8 Tanaka, Y., Closed images of locally compact spaces and Fréchet spaces, Topology Proceedings, 7 (1982), 279~292.

9 Tanaka, Y., Metrization II, in. K. Morita, J. Nagata Eds., Topics in General Topology (Elsevier Science Publishers B. V., 1989), 275~314.

局部紧度量空间的闭映象

林　寿

(宁德师专,352100,福建)

摘　要　本文的主要结果是 T_2 空间 X 具有 σ 遗传闭包保持的紧 k 网当且仅当 X 具有 σ 遗传闭包保持的闭 k 网并且 X 的每一闭度量子空间是局部紧。作为它的应用,我们建立了局部紧度量空间的闭映象的新特征。

关键词　k 网,局部紧空间,度量空间,闭映射

分类号　54C10, 54E99

南京大学学报数学半年刊
第12卷 第1期 JOURNAL OF NANJING UNIVERSITY Vol. 12, No. 1
1995年5月 MATHEMATICAL BIQUARTERLY May, 1995

THE STABILIZATION OF SEMIGROUP OF CLASS (1,A)*

Xue Xingmei
(Southeast University)

Abstract Let $S(t)$;$t \geqslant 0$ be a semigroup of class $(1,A)$ in Banach space X, with infinitesimal generator A. In this paper we prove that: if either $S(t)$ or $S^*(t)$ is strongly stable, yet not uniformly exponentially stable as $t \to +\infty$, then for any compact operator B on X, the semigroup $S^B(t)$ generated by $A+B$ can't be uniformly exponentially stable as $t \to \infty$.

Keywords semigroup of class $(1,A)$; strongly stable; uniformly exponentially stable

Let $S(t), t \geqslant 0$ be a semigroup in Banach space $(X, \| \cdot \|)$ and A be its infinitesimal generator. Let $B \in B(X)$, the Banach space of bounded linear operators on X, we denote by $S^B(t), t \geqslant 0$, the semigroup generated by $A+B$, which corresponds with the abstract linear dynamic system:

$$x'(t) = Ax(t) + U$$

with U in feedback form $U = Bx(t)$, whose solution is given by $x(t, x_0) = S^B(t) x_0$.

Feedback control problems of this type arise from control theoretic studies for linear dynamical systems ,where an aim is to select the feedback operator B as to force $S^B(t)$ to posses asymptotic stability properties. Otherwise it can be used to discuss the positive solutions of the algebraic Riccatic equations(see[1]).

Definition 1 Let $S(t), t \geqslant 0$ be a semigroup of linear operators, it is a semigroup of class $(1,A)$ if it satisfies:

1) for any $x \in X$, $t_0 > 0$, $\lim_{t \to t_0} S(t)x = S(t_0)x$,

2) for any $x \in X$, $\lim_{\lambda \to +\infty} \lambda R(\lambda)x = \lim_{\lambda \to +\infty} \lambda \int_0^\infty e^{\lambda t} S(t) x \mathrm{d}t = x$, and

$$\int_0^1 \| S(t) \| \mathrm{d}t < +\infty,$$

where $R(\lambda) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$.

---

* Research supported by the National Natural Science Foundation of China.
收稿日期:1994—05—05.