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THE BARRELLED PROPERTY OF FUNCTION SPACES $C_p(Y|X)$ AND $C_k(Y|X)$

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1. Introduction

For a completely regular T_1 space X, $C(X)$ denotes the set of all continuous real-valued functions on X. If Y is a subspace of X, $C(Y|X) = \{f \in C(Y) : f$ is extendable continuously over $X\}$. $C_p(Y|X)$, topologized as a subspace of $C_p(Y)$, has proved to be useful in the theory of function spaces. For example, it was used by Lutzer and McCoy in [5] to characterize the Baire property of $C_p(X)$ as "$C_p(X)$ is Baire if and only if for each countable $Y \subseteq X$, the space $C_p(Y|X)$ is Baire". The barrelled property is an important topic of functional analysis. It is connected to general topology via the Nachbin-Shirota theorem of function spaces which characterizes the barrelled property of $C_k(X)$ topologically. In the first part of this paper, we give topological characterizations of the barrelled property of $C_p(Y|X)$ and $C_k(Y|X)$ in the similar form of Nachbin-Shirota theorem. As an application of the main results, we obtain a paralled characterization of the barrelled property of $C_p(X)$ to that of the Baire property quoted above. Since $C_p(Y|X)$ and $C_k(Y|X)$ are dense in $C_p(Y)$ and $C_k(Y)$ respectively, the Baire property and the barrelled property of the former imply that of the latter. So it is an interesting question (see Question 3.9 in [5]) that whether the Baire property of $C_p(X)$ can be characterized by using $C_p(Y)$ instead of $C_p(Y|X)$. We give an example to show that the answer to this question restated for barrelled property in place of the Baire property is negative. All the applications and examples are put in the second part.

Let X be a space and $Y \subseteq X$. For $A \subseteq Y$, $f \in C(Y|X)$ and $\epsilon > 0$, denote $W[f, A, \epsilon;\ Y|X] = \{g \in C(Y|X) : | f(x) - g(x) | \leq \epsilon$ for $x \in A\}$ and $W(f, A, \epsilon;\ Y|X) = \{g \in C(Y|X) : | f(x) - g(x) | < \epsilon$ for $x \in A\}$. $\{W(f, F, \epsilon;\ Y|X) : f \in C(Y|X)$, $F \subseteq Y$ is finite and $\epsilon > 0\}$ and $\{W(f, K, \epsilon;\ Y|X) : f \in C(Y|X), K \subseteq Y$ is compact and $\epsilon > 0\}$ are bases of $C_p(Y|X)$ and $C_k(Y|X)$ respectively. In the case of $Y = X$, $W[f, A, \epsilon;\ Y|X]$ and $W(f, A, \epsilon;\ Y|X)$ are denoted as $W[f, A, \epsilon]$ and $W(f, A, \epsilon)$ respectively.

A set $A \subseteq X$ is called bounded in X(or a bounded set of X) if for each $f \in C(X)$, the restriction $f \restriction_A$ of f on A is bounded. Let E be a linear topological space. A set $B \subseteq E$ is called convex if $ta + (1 - t)b \in B$ for $t \in [0, 1]$ and $a, b \in B$. B is called circled if $ta \in B$ for $t \in [-1, 1]$ and $a \in B$. If for each $x \in E$, there exists $\lambda_x > 0$ such that $[0, \lambda_x]x \subseteq B$, then B is said absorbent. B is called a barrel if it is closed, convex, circled and absorbent. A linear topological space E is called locally convex if it has a neighborhood base of zero consisting of convex sets. Note that $C_p(X), C_k(X)$ and every normed linear space are locally convex. For the definition of normed linear space, refer to any standard textbook of functional analysis. A complete normed linear space is called a Banach space. Let $C^*(X)$ denote all the bounded continuous functions on X and $C^*_n(X)$ the normed space with the supremum norm. Then $C^*_n(X)$ is a Banach space. A locally convex linear topological space is called barrelled (or has barrelled property) if each barrel is a neighborhood of zero. A topological space is Baire if the intersection of countably many open dense sets of the space is dense in it. It is known that each Banach space is Baire and each Baire locally convex linear topological space is barrelled.

The following theorems are well-known.

Theorem I. *(Nachbin and Shirota, see [6] or [7]).* For a space X, the following are equivalent:

(1) Every bounded closed set of X is compact.
(2) $C_k(X)$ is barrelled.

Theorem II. *(Buchwalter and Schmets [2]).* For a space X, the following are equivalent:
(1) Every bounded set of X is finite.
(2) $C_p(X)$ is barrelled.

We assume all spaces in this paper are completely regular T_1.

2. Characterizations for the barrelled property of $C_p(Y|X)$ and $C_k(Y|X)$

Let X be a space. For a subset H of $C(X)$, let

$$K(H) = \{x \in X : \text{for every neighborhood } U \text{ of } x \text{ there is } f \notin H \text{ such that } f|_{X\setminus U} = 0\}.$$

Obviously, $K(H)$ is closed in X.

The result below is useful.

Proposition 2.1 *(Asanov and Shamgunov [1]).* If H is a barrel in $C_k(X)$, then $K(H)$ is bounded in X.

To prove the characterization theorems we give two lemmas first.

Lemma 2.2. Let L be a compact set of X. If $\{U_1, U_2, \ldots, U_n\}$ is a collection of open sets of X covering L, then there are $h_1, h_2, \ldots, h_n \in C(X, [0,1])$ such that for each $i, h_i^{-1}(0,1]) \subseteq U_i$, $h = \sum_{i=1}^{n} h_i \in C(X, [0,1])$ and $h(x) = 1$ for each $x \in L$.

Proof: Choose a partition of unity $\{f_0, f_1, \ldots, f_n\}$ of X subordinate to the cover $\{U_0, U_1, \ldots, U_n\}$, where $U_0 = X \setminus L$. Let $h_i = f_i$ for $i \geq 1$. Then h_i's are desired.

$C^*(X)$ denotes all the bounded continuous functions on X and $C^*(Y|X) = C^*(Y) \cap C(Y|X)$. $C_n^*(X)$ and $C_n^*(Y|X)$ are the space $C^*(X)$ and $C^*(Y|X)$ respectively with the uniform norm topology. It is easy to see that $C_p(Y|X)(C_k(Y|X))$ is dense in $C_p(Y)(C_k(Y)$ resp.). This is different for $C_n^*(Y|X)$, because

Lemma 2.3. $C_n^*(Y|X)$ is a closed subspace of $C_n^*(Y)$.

Proof: denote $G = \{f \in C^*(X) : f|_Y = 0\}$. Then G is a closed linear subspace of $C_n^*(X)$. Let $C_n^*(X)/G$ be the quotient space of the normed space $C_n^*(X)$ mod G. Since $C_n^*(X)$ is a Banach space and G is closed, $C_n^*(X)/G$ is a Banach space. It can be proved that there is a norm-preserving linear mapping between $C_n^*(Y|X)$ and $C_n^*(X)/G$. So $C_n^*(Y|X)$ is also a Banach space. $\square$.

Theorem 2.4. for a space X and a subspace Y of it, the following are equivalent:

(1) $C_k(Y|X)$ is barrelled.
(2) Every closed subset of Y which is bounded in X is compact.

Proof: The idea of this proof is similar to that of [1].

(1) $\to$ (2). Let A be a closed subset of Y which is bounded in X. It is easy to see that $W[f_0, A, 1;\ Y|X]$ is a barrel in $C_k(Y|X)$, where $f_0 = 0$ By (1), it is a neighborhood of f_0 and so there are a compact set K of Y and $\epsilon > 0$ such that $W(f_0, K, \epsilon;\ Y|X) \subseteq W[f_0, A, 1;\ Y|X]$. This implies that $A \subseteq K$ and so A is compact.

(2) $\to$(1). Let $\pi_Y : C_k(X) \to C_k(Y|X)$ be the restriction mapping. Then π_Y is a continuously linear mapping. Given a barrel H in $C_k(Y|X)$, then $\pi_Y^{-1}(H)$ is a barrel in $C_k(X)$. By Proposition 2.1, $K(\pi_Y^{-1}(H))$ is a bounded closed set of X. Let $G(H) = K(\pi_Y^{-1}(H)) \cap Y$. Then $G(H)$ is a closed set of Y which is bounded in X. By assumption, $G(H)$ is compact. We proved that there is $\epsilon > 0$ such that $W(f_0, G(H), \epsilon;\ Y|X) \subseteq H$. At first, we prove

Claim. If $f \in C(Y|X)$ and $f|_{G(H)} = 0$, then $f \in H$.

Proof of Claim: Suppose $f \notin H$. Since H is closed in $C_k(Y|X)$, there are a compact set K of Y and $\rho > 0$ such that $W(f, K, \rho;\ Y|X) \cap H = \emptyset$. We may assume that $G(H) \neq Y$, otherwise $C_k(Y|X)$ is just the barrelled space $C_n^*(Y)$. Since $f|_{G(H)} = 0$, there is an open set U of X such that $K \cap G(H) \subseteq U$

and for each $x \in U \cap Y, |f(x)| < \rho$. Denote $L = K \backslash U$. By the definition of $G(H)$ and the compactness of L, there are open sets $V_1, V_2, \ldots, V_n$ of X such that $L \subseteq_{i=1}^{n} V_i$ and for each i, if $h \in C(X)$ and $h|_{X \backslash V_i} = 0$, then $h \in \pi_Y^{-1}(H)$. By Lemma 2.2, there are $h_1, h_2, \ldots, h_n \in C(X, [0,1])$ such that $h_i|_{X \backslash V_i} = 0$, $h = \sum_{i=1}^{n} h_i \in C(X, [0,1])$ and $h(x) = 1$ for every $x \in L$. Let $\tilde{f}$ be a continuous extension of f over X and $\tilde{g} = \sum_{i=1}^{n} h_i \tilde{f}$. Since $nh_i \tilde{f}|_{X \backslash V_i} = 0$, $nh_i \tilde{f} \in \pi_Y^{-1}(H)$. As $\pi_Y^{-1}(H)$ is convex, $\tilde{g} = \sum_{i=1}^{n} (1/n)(nh_i \tilde{f}) \in \pi_Y^{-1}(H)$. Let $g = \tilde{g}|_Y$. Then $g \in H$. Now we derive a contradiction from proving that $g \in W(f, K, \rho;\ Y|X)$, i.e., $\mid f(x) - g(x) \mid < \rho$ for every $x \in K$. As $f|_L = g|_L$, we may assume $x \in K \cap U$. Then $\mid f(x) = f(x) \mid = \mid 1 - \sum_{i=1}^{n} h_i(x) \mid \mid f(x) \mid \leq \mid f(x) \mid < \rho$, and the claim is proved.

By Lemma 2.3, $C_n^*(Y|X)$ is a Banach space. Because $H \cap C_n^*(Y|X)$ is a barrel in $C_n^*(Y|X)$, there exists $\epsilon > 0$ such that $W(f_0, Y, 3\epsilon;\ Y|X) \subseteq H \cap C_n^*(Y|X)$. We prove that this ϵ is the required one.

Let $f \in W(f_0, G(H), \epsilon;\ Y|X)$. Define $g(x) = \max\{f(x), \epsilon\} + \min\{f(x), -\epsilon\}$. Then $g \in C(Y|X)$, $g|_{G(H)} = 0$ and $\mid f(x) - g(x) \mid \leq \epsilon$ for each $x \in Y$. IT follows that $2g \in H$ by the claim and $2(f-g) \in H \cap C_n(Y|X)$. Since $f = (1/2)(2g) + (1/2)(2f - 2g)$, $f \in H$. This completes the proof of the theorem. □

Remark. If L in the proof is empty, let $g = f_0$, the zero function.

Theorem 2.5. For a space X and a subspace Y of it, the following are equivalent:

(1) $C_p(Y|X)$ is barrelled.

(2) Every (closed) subset of Y which is bounded in X is finite.

Proof: (1) $\rightarrow$ (2). Assume A is a subset of Y which is bounded in X. Then $W[f_0, A, 1;\ Y|X]$ is a barrel in $C_p(Y|X)$. By the assumption, there are a finite subset F of Y and $\epsilon > 0$ such that $W(f_0, F, \epsilon;\ Y|X) \subseteq W[f_0, A, 1;\ Y|X]$. This implies that $A \subseteq F$ and thus A is finite.

(2) $\to$ (1). Let H be a barrel in $C_p(Y|X)$. Then $\pi_Y^{-1}(H)$ is a barrel in $C_p(X)$ and thus in $C_k(X)$. By Proposition 2.1, $K(\pi_Y^{-1}(H))$ is bounded in X. Let $G(H)$ is finite. By the same method as in the proof of the foregoing theorem, it is proved that there is $\epsilon > 0$ such that $W(f_0, G(H), \epsilon;\ Y|X) \subseteq H$. □

3. Applications and examples

At first, we derive some results from the theorems in §2.

Proposition 3.1. For a space X and a subspace Y of it. if $C_p(X)$ is barrelled, then $C_p(Y|X)$ is barrelled.

Proposition 3.2. For a space X and a closed subspace Y of it, $C_k(X)$ is barrelled, then $C_k(Y|X)$ is barrelled.

These follow directly from Theorem I and II in §1 and Theorem 2.4 and 2.5.

Theorem 3.3. For a space $X, C_p(X)$ is barrelled iff for every countable subspace Y of $X, C_p(Y|X)$ is barrelled.

Proof: Necessity follows from Proposition 3.1. Now assume that for every countable subspace Y of $X, C_p(Y|X)$ is barrelled. If $C_p(X)$ is not barrelled, by Theorem II, there is an infinite bounded set A in X. Let Y be a countably infinite subset of A. Then Y is bounded in X but not finite. By Theorem 2.5, $C_p(Y|X)$ is not barrelled. This contradiction show that $C_p(X)$ is barrelled. □

As $C_p(Y|X)(C_kY|X))$ is dense in $C_p(Y)(C_k(Y)$ resp.), if $C_p(Y|X)(C_k(Y|X))$ is barrelled, then $C_p(Y)C_k(Y)$ reap.) is barrelled. The following examples shows that neither converse holds. It also emphasizes that Proposition 3.2 needs Y to be closed and that "k" may not replace "p" in Theorem 3.3

Example 3.4. Let $X = \{0\} \cup \{1/n : n \in N\}$ with the subspace topology of R and $Y = \{1/n : n \in N\}$. Then $C_k(X) = C_n^*(X)$ and $C_p(Y) = R^\omega = C_k(Y)$ are barrelled, but $C_p(Y|X) = C_k(Y|X)$ is not barrelled.

Proof: $C_p(Y|X) = C_k(Y|X)$ is not barrelled because Y is bounded in X but neither finite nor compact. □

Let E and F be linear topological spaces and f an open linear continuous mapping from E onto F. Then if E is barrelled, F is also barrelled. Although $\pi_Y : C_n^*(X) \to C_n^*(Y|X)$ is always an open mapping (Under this map the unit ball of $C_n^*(Y|X)$ is the image of the unit ball of $C_n^*(X)$), $\pi_Y : C_p(X) \to C_p(Y|X)$ (or $\pi_Y : C_k(X) \to C_k(Y|X)$) need not be (see Example 3.4). However, as the following shows, they are open when Y is a closed subspace of X. So Proposition 3.2 can be also proved in this way.

Proposition 3.5. Let X be a space and Y a closed subspace of X. Then $\pi_Y : C_p(X) \to C_p(Y|X)$ and $\pi_Y : C_k(X) \to C_k(Y|X)$ are open mappings.

Proof: We only give a proof for the case of "k", that for the case of "p" is similar. Let $W(f, K, \epsilon)$ be an open set of $C_k(X)$, where $f \in C(X), K$ is a compact set of X and $\epsilon > 0$. Then $\pi_Y(W(f, K, \epsilon)) = W(f|_Y, K \cap Y, \epsilon;\ Y|X)$. In fact, it is easy to see that $\pi_Y(W(f, K < \epsilon)) \subseteq W(f|_Y, K \cap Y, \epsilon;\ Y|X)$. Let $g \in W(f|_Y, K \cap Y, \epsilon;\ Y|X)$. Choose a continuous extension $\tilde{g}$ of g and then an open set U of X such that $K \cap Y \subseteq U$ and $\sup\{| f(x) - \tilde{g}(x) |: x \in U\} < \epsilon$. Since $K \backslash U$ and Y are disjoint closed sets of X and $K \backslash U$ is compact, there is $\phi \in C(X, 0, 1)$ such that $\phi(K \backslash U) = 0$ and $\phi(Y) = 1$. Let $\tilde{\tilde{g}} = \tilde{g} \cdot \phi + f \cdot (1 - \phi)$. Then $\tilde{\tilde{g}}|_Y = \tilde{g}|_Y = g$ and

$$\begin{aligned}
&\sup \{| f(x) - \tilde{\tilde{g}}(x) |: x \in K\} \\
&= \sup\{| f(x) - \tilde{g}(x) \cdot \phi(x) - f(x) + f(x) \cdot \phi(x) |: x \in K\} \\
&= \sup\{| f(x) - \tilde{g}(x) | \cdot | \phi(x) |: x \in K\} \\
&= \max\{\sup\{| f(x) - \tilde{g}(x) | \cdot | \phi(x) |: x \in K \backslash U\}, \\
&\qquad \sup\{| f(x) - \tilde{g}(x) | \cdot | \phi(x) |: x \in K \cap U\}\}
\end{aligned}$$

$= \sup\{\mid f(x) - \tilde{g}(x) \mid \cdot \mid \phi(x) \mid: x \in K \cap U\}$
$\leq \sup\{\mid f(x) - \tilde{g}(x) \mid: x \in K \cap U\} < \epsilon.$
So $\tilde{\tilde{g}} \in W(f, K, \epsilon)$ and thus $g \in \pi_Y(W(f, K, \epsilon))$. The proof is complete. □

Let X and Y be the spaces in Example 3.4. $W(f_0, \{0\}, 1)$ is an open set of $C_p(X)$, where $f_0 = 0$. Then $\pi_Y(W(f_0, \{0\}, 1))$ is the set of all convergent sequences with limits in $(-1, 1)$. It is obvious that $\pi_Y(W(f_0, \{0\}, 1))$ is not open in $C_k(Y|X)$.

In an opposite direction to Example 3.4, we give an example to show that an analog for $C_k(X)$ of Theorem 3.3 is not true.

Example 3.6. Let $X = \omega_1$ with the interval topology. Then for every countable closed subspace Y of $C_k(Y|X)$ is barrelled, but $C_k(X)$ is not.

Proof: Since every countable closed subspace Y of X is compact, so $C_k(Y|X) - C_k(Y) = C_n^*(Y)$ is barrelled. However, since X is a pseudo-compact but not compact space. By Theorem I, $C_k(X)$ is not barrelled. □

It is shown in [4] that for a space X and a subspace Y of it, $C_p(Y|X)$ is Čech - complete if and only if $C_p(Y)$ is Čech - complete and Y is C - embedded in X, i.e., every continuous function on Y can be continuously extended over X. But this is different for the barrelled property.

Example 3.7. Let $X = \omega \cup \{p\}$ and $Y = \omega$, where $p \in \beta\omega \backslash \omega$. Then $C_p(Y|X)$ and $C_p(Y) = R^\omega$ are barrelled, but Y is not C - embedded in X.

Proof: Note that if Y is countable and C - embedded in X, then Y must be closed in X. So Y is not C - embedded in X. To prove that $C_p(Y|X)$ is barrelled ($C_p(Y)$ is obviously barrelled), we prove that every countably infinite subset A of y must be unbounded in X. Let $A = A_1 \cup A_2$ such that $|A_1| = \aleph_0 = |A_2|$ and $A_1 \cap A_2 = \emptyset$. Since $\mathrm{cl}_X A_1 \cap \mathrm{cl}_X A_2 = \emptyset$, without loss of generality, we may assume that $p \notin \mathrm{cl}_X A_1$. Denote $A_1 = \{a_n :$

$n \in N\}$. Define a continuous function $f \in C(X)$ as $f(a_n) = n$ and $f(x) = 0$ for other x's in X. Then $f|_A$ is unbounded.

At last, we give an example to show that in Theorem 3.3, $C_p(Y|X)$ can not be replaced with $C_p(Y)$. This also answers a question in [5] (Question 3.9) restated for the barrelled property in place of the Baire property.

Example 3.8. Let $X = \beta\omega$. then for every countable subspace Y of X, $C_p(Y)$ is barrelled, but $C_p(X)$ is not barrelled.

Proof: It is obvious by Theorem II that $C_p(X)$ is not barrelled. Let Y be a countable subspace of X. The following fact about $\beta\omega$ is used now and then in the proof:

"If $x \in \beta\omega$ and $B \subseteq \beta\omega$ with $|B| = \aleph_0$, then there is a clopen neighborhood U of x such that $|B \backslash U| = \aleph_0$".

We can assume that Y is infinite, otherwise, $C_p(Y)$ is obviously barrelled. Let A be an infinite subset of Y. We prove that A is unbounded in Y. Then by Theorem II, $C_p(Y)$ is barrelled.

Enumerate Y as $\{y_i : i \in N\}$. Let $n_1 = 1$. By the fact above, there is a clopen neighborhood U_1 of y_{n_1} such that $|A \backslash U_1| = \aleph_0$. Let $n_2 = \min\ \{i : i \in N$ and $y_i \notin U_1\}$. Then there is a clopen neighborhood U_2 of y_{n_2} such that $U_1 \cap U_2 = \emptyset$ and $|A \backslash (U_1 \cup U_2)| = \aleph_0$. Let $n_3 = \min\{i : i \in N$ and $y_i \notin U_1 \cup U_2\}$. There is a clopen neighborhood U_3 of y_{n_3} such that $U_3 \cap (U_1 \cup U_2) = \emptyset$ and $|A \backslash (U_1 \cup U_2 \cup U_3)| = \aleph_0$. Continuing in this way, we obtain a pairwise disjoint cover $\{U_i : i \in N\}$ of Y by clopen sets of X such that A has nonempty intersections with infinitely many members of the cover. Let $W_i = U_i \cap Y$ for $i \in N$. Then $\{W_i : i \in N\}$ is a pairwise disjoint open cover of Y. Define a function $f \in C(Y)$ as $f(y) = i$ if and only if $y \in W_i$. Then f is obviously unbounded on A. The proof finishes. □

Remark. The authors would like to thank the referee for simplifying the proof of Example 3.8. The original proof involves a case discussion for the derived set of Y.

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