Filomat 32:17 (2018), 6115–6129
https://doi.org/10.2298/FIL1817115L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Spaces Defined by Pytkeev Networks

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Abstract. The notions of networks and k-networks for topological spaces have played an important role in general topology. Pytkeev networks, strict Pytkeev networks and cn-networks for topological spaces are defined by T. Banakh, and S. Gabriyelyan and J. Kąkol, respectively. In this paper, we discuss the relationship among certain Pytkeev networks, strict Pytkeev networks, cn-networks and k-networks in a topological space, and detect their operational properties.

It is proved that every point-countable Pytkeev network for a topological space is a quasi-k-network, and every topological space with a point-countable cn-network is a meta-Lindelöf D-space, which give an affirmative answer to the following problem [25, 29]: Is every Fréchet-Urysohn space with a point-countable cs'-network a meta-Lindelöf space? Some mapping theorems on the spaces with certain Pytkeev networks are established and it is showed that (strict) Pytkeev networks are preserved by closed mappings and finite-to-one pseudo-open mappings, and cn-networks are preserved by pseudo-open mappings, in particular, spaces with a point-countable Pytkeev network are preserved by closed mappings.

1. Introduction

As we know, metrization theory is the core in the study of general topology, and the theory of generalized metric spaces is an important generalization of this theory [19, 28]. The ideas and problems of generalized metric spaces, in particular, the general metrization problem, greatly influenced all domains of set-theoretic topology [22]. The Bing-Nagata-Smirnov metrization theorem evoked enthusiasm on spaces with various base-like properties, for example, networks, weak bases, k-networks and cs-networks, etc. On the other hand, a study of countability is an important task on general topology. In order to generalize certain countability, all kinds of spaces put forward by topologists, like Lindelöf spaces, Fréchet-Urysohn spaces, sequential spaces and countable tightness, etc.

In 1983, E.G. Pytkeev [39] proved that every sequential space satisfies the following property, known actually as the Pytkeev property [30, Definition 0.2], which is stronger than countable tightness: a topological space X has the *Pytkeev property* if for each $A \subset X$ and each $x \in \overline{A} \setminus A$, there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of infinite subsets of A such that each neighborhood of x contains some A_n. B. Tsaban and L. Zdomskyy [45, Definition 5] strengthened this property as follows: a topological space X has the *strong Pytkeev property*

2010 *Mathematics Subject Classification*. Primary 54C10; Secondary 54D20, 54E20
Keywords. Networks; Pytkeev networks, strict Pytkeev networks, cn-networks, k-networks, point-countable families, meta-Lindelöf spaces, D-spaces, closed mappings, pseudo-open mappings
Received: 21 April 2018; Revised: 09 September 2018; Accepted: 14 September 2018
Communicated by Ljubiša D.R. Kočinac
Research supported by the NSFC (No. 11471153, 11801254)
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if for each $x \in X$, there exists a countable family $\mathcal{P}$ of subsets of X, such that for each neighborhood U of x and each $A \subset X$ with $x \in \overline{A} \setminus A$, there is a set $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap A$ is infinite. Clearly, the strong Pytkeev property implies countable cs^*-character [45, p. 8]. These lead people to research the relation among the strong Pytkeev property, countability and generalized metric properties.

In 2015, the notions of Pytkeev networks and strict Pytkeev networks were introduced [4]. T. Banakh [6, Proposition 1.8] proved that each countable (strict) Pytkeev network is a k-network (resp. cs^*-network) in a topological space, and the converse is also true for a k-space (resp. Fréchet-Urysohn space). At the same time, the notions of cp-networks, ck-networks and cn-networks were introduced [15]. S. Gabriyelyan and J. Kąkol [15, Proposition 2.3] proved each space with a countable cn-network at each point is of countable tightness. Various topological spaces have been defined by spaces with certain Pytkeev networks, strict Pytkeev networks, cp-networks, ck-networks and cn-networks, for example, $\mathfrak{P}_0$-spaces [4], $\mathfrak{P}$-spaces [15], strict σ-spaces [15] and strict $\aleph$-spaces [15], which played an important role in generalized metric spaces, cardinal functions, hyperspaces, function spaces, topological groups and topological vector spaces [4–6, 14–17, 23, 45].

It was a problem that "It is natural to ask for which topological spaces some of the types of networks coincide" [15, p. 182]. Partial answers to this problem were given by T. Banakh [4], and S. Gabriyelyan and J. Kąkol [15], etc. They studied the relationship among countable families or σ-locally finite families with certain Pytkeev networks, k-networks and cs^*-networks. The following problem is discussed continuously in this paper.

Problem 1.1. *Find relationship between some classes of networks in topological spaces.*

It is well known that each k-network or each cs^*-network for a topological space is a wcs^*-network [27], and each cs^*-network for a topological space is a cs'-network [26]. These guide us to discuss the relationship among Pytkeev networks, wcs^*-networks and cs'-networks for a topological space. On the other hand, point-countable families of subsets are an essential generalization of σ-locally finite families in topological spaces. In particular, spaces with a point-countable base have some interesting properties. For example, spaces with a point-countable base are a meta-Lindelöf D-space [3, Theorem 2], and spaces with a point-countable base are preserved by perfect mappings [12]. Spaces with certain point-countable families have also some similar properties. G. Gruenhage, E.A. Michael and Y. Tanaka [21, Proposition 8.6] proved that regular Fréchet-Urysohn spaces with a point-countable k-network are a meta-Lindelöf space. L.X. Peng [37, Corollary 22] proved that regular k-spaces with a point-countable k-network are a D-space. D.K. Burke [9, Corollary 4.5] proved that spaces with a point-countable weak base are a D-space. S. Lin and Z.Y. Cai [26, Lemma 2.7] proved that sequential spaces with a point-countable cs'-networks are a D-space. The following problems are interesting.

Problem 1.2. ([25, Question 2.1.24][29, Question 2.1]) *Is every Fréchet-Urysohn space with a point-countable cs'-network a meta-Lindelöf space?*

Problem 1.3. ([38, p. 1050]) *Is every sequential space with a point-countable wcs^*-network a D-space?*

These motivate us to study the spaces with certain Pytkeev networks, in particular, to discuss further spaces with certain point-countable Pytkeev networks. Operations over spaces with certain Pytkeev networks were discussed [4, 6, 15], which contained hereditary properties, topological sum properties, Tychonoff product properties and box product properties, etc. Mapping properties on spaces with certain Pytkeev networks are seldom discussed in the literature [30, 46]. It was proved that the Pryteev property is preserved by a closed mapping [46, Theorem 1], and the Pryteev property is inversely preserved by a perfect mapping with the fibers of countable tightness [30, Theorem 5.4]. E.A. Michael [32, Theorem 11.4] proved that a regular space with a countable k-network (i.e., an $\aleph_0$-space) if and only if it is a compact-covering image of a separable metrizable space; and Z.W. Li [24, Theorem 4] proved that a regular space is with a σ-locally finite k-network (i.e., an $\aleph$-space) if and only if it is a sequence-covering *mssc*-image of a metrizable space.

Problem 1.4. ([15, Question 6.8]) *Find a characterization of regular spaces with a countable strict Pytkeev network analogous to the characterization of $\aleph_0$-spaces given by E.A. Michael.*

Problem 1.5. ([15, Question 6.10]) *Characterize regular spaces with a σ-locally finite ck-network and regular spaces with a σ-locally finite strict Pytkeev networks analogous to the characterization of $\aleph$-spaces given by Z.W. Li.*

The paper is organized as follows. In Section 2, we introduce some notions to be discussed in this paper, and describe some basic relation among spaces defined by these notions. In Section 3, we discuss Problem 1.1 and obtain the relationship among spaces with certain networks, which involve Pytkeev networks, strict Pytkeev networks, *cn*-networks, *k*-networks, *wcs**-networks and *cs*$'$-networks, prove that a space with a point-countable Pytkeev network is metrizable if and only if it is an *M*-space. In Section 4, we detect the covering properties on spaces with certain *cn*-networks and prove that every space with a point-countable *cn*-network is a meta-Lindelöf D-space, which gives an affirmative answer to Problem 1.2 and a partial answer to Problem 1.3. In Section 5, we establish some mapping theorems on spaces with certain Pytkeev networks and prove that closed mappings or finite-to-one pseudo-open mappings preserve (strict) Pytkeev networks, and pseudo-open mappings preserve *cn*-networks, which will be beneficial to the solution to Problems 1.4 and 1.5.

2. Preliminaries

In this section, we introduce the necessary notation, terminology, and describe some basic relation among spaces defined by these notions. Throughout this paper, all topological spaces are assumed to be Hausdorff, mappings are continuous and onto, unless explicitly stated otherwise. The sets of real numbers, rational numbers and positive integers are denoted by $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$, respectively.

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a topological space X.

(1) $\mathcal{P}$ is a *network* [11, p. 127] for X, if for any neighborhood U of a point $x \in X$, there exists a set $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is a *k-network* [36, Definition 1] (resp. *quasi-k-network* [21, p. 310]) for X, if whenever K is a compact (resp. countably compact) subset of an open set U in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

(3) $\mathcal{P}$ is a *cs*-network* [18, Definition 3] (resp. *wcs*-network* [27, p. 79]) for X, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in U$ with U open in X, there exists a set $P \in \mathcal{P}$ such that some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P and $x \in P \subset U$ (resp. $P \subset U$).

(4) $\mathcal{P}$ is a *cs$'$-network* [26, p. 1065] for X, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in U$ with U open in X, there exists a set $P \in \mathcal{P}$ such that $\{x, x_m\} \subset P \subset U$ for some $m \in \mathbb{N}$.

(5) $\mathcal{P}$ is a *Pytkeev network* (resp. *strict Pytkeev network*) [4, Definition 1.1] for X, if $\mathcal{P}$ is a network for X, and for each neighborhood U of a point x in X and each subset A of X accumulating at x, there exists a set $P \in \mathcal{P}$ such that $P \cap A$ is infinite and $P \subset U$ (resp. $x \in P \subset U$).

(6) $\mathcal{P}$ is a *cn-network* [15, Definition 1.1] for X, if for each neighborhood U of a point x in X, the set $\bigcup\{P \in \mathcal{P} : x \in P \subset U\}$ is a neighborhood of x.

For a fixed point $x \in X$, a family $\mathcal{P}$ of subsets of X is called a *network* (resp. *cs*-network* (*wcs*-network*), *cs$'$-network*, *Pytkeev network* (*strict Pytkeev network*) or *cn-network*) of the point x in X, if the family $\mathcal{P}$ satisfies the above mentioned conditions (1) (resp. (3), (4), (5) or (6)) at x.

Remark 2.2. (1) In [4, p. 152], it was said that a subset A of a topological space X accumulates at a point $x \in X$ if each neighborhood of x contains infinitely many points of the set A. It is obvious that for a T_1 space X a point $x \in X$ is an accumulation point of a set $A \subset X$ if and only if $x \in \overline{A \setminus \{x\}}$.

(2) The notion of *cp*-networks is introduced in [15, Definition 1.1]. A family $\mathcal{P}$ of subsets of a space X is called a *cp-network* at a point $x \in X$ if either x is an isolated point of X and $\{x\} \in \mathcal{P}$, or for each subset A of X with $x \in \overline{A} \setminus A$ and each neighborhood U of x, there exists a set $P \in \mathcal{P}$ such that $P \cap A$ is infinite and $x \in P \subset U$. It is easy to check that for a T_1 space X a family $\mathcal{P}$ of subsets of X is a strict Pytkeev network if and only if it is a *cp*-network for X.

Let X be a topological space and $A \subset X$. The set A is *k-closed* in X if $A \cap K$ is relatively closed in K for each compact subset K of X. A space X is a *k-space* [11, p. 152] if every k-closed set of X is closed.

Lemma 2.3. ([4, Proposition 1.7]) *Each k-network in a k-space is a Pytkeev network.*

We will show that each *cn*-network for a space is a *cs*′-network, see Corollary 3.8. The following figure is some basic relationship among spaces with certain networks to be discussed in this paper.

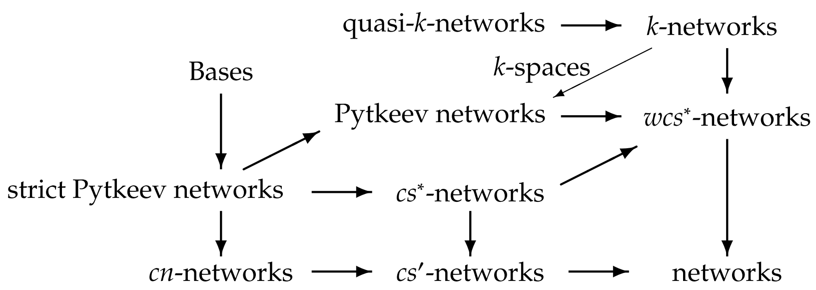

Figure 1 The relationship among spaces with certain networks

It is easy to see that every base for a space is a k-network, but a base for a space is not always a quasi-k-network. Let $X = [0, \omega_1)$, and it be endowed with the ordered topology. Put $\mathcal{P} = \{U \subset X : U$ is open and $U \subset [0, \alpha)$ for some $\alpha < \omega_1\}$. Then $\mathcal{P}$ is a base for X. Since X is a countably compact space and X is not covered by any finitely many elements of $\mathcal{P}$, $\mathcal{P}$ is not a quasi-k-network for X.

By the notions of certain networks, some generalized metrizable spaces were defined, which have played an important role in the study of general topology.

Definition 2.4. Let X be a regular space.

(1) The space X is a *cosmic space* [32, p. 993] (resp. an *$\aleph_0$-space* [32, Definition 1.2], a *$\mathfrak{P}_0$-space* [4, Definition 1.2]) if it has a countable network (resp. k-network, Pytkeev network).

(2) The space X is a *σ-space* [35, p. 236] (resp. a *strict σ-space* [15, Definition 1.5], an *$\aleph$-space* [36, Definition 2], a *strict $\aleph$-space* [15, Definition 1.5], *$\mathfrak{P}$-space* [15, Definition 1.4]) if it has a σ-locally finite network (resp. *cn*-network, k-network, *ck*-network[1)], strict Pytkeev network).

The following figure describes the relation among spaces defined by certain networks [15, p. 180].

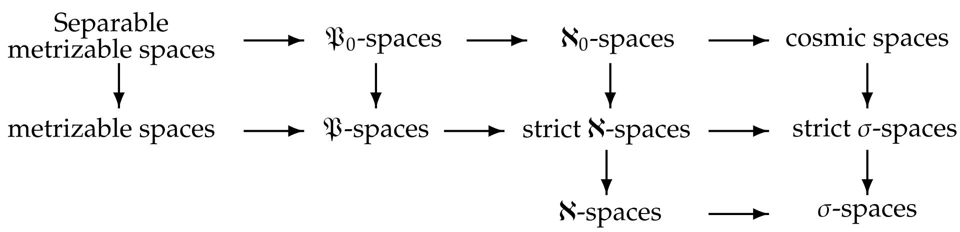

Figure 2 The relationship among spaces defined by certain networks

[1)]The notion of *ck*-networks does be discussed in this paper. A family $\mathcal{P}$ of subsets of a topological space X is called a *ck-network* [15, Definition 1.1] for X if, for any neighborhood U of a point x in X, there is a neighborhood U_x of x such that for each compact subset K of U_x there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset U$.

The further relationship among spaces with certain networks involves weakly first-countability. Recall some related notions. Let X be a topological space and $A \subset X$. The set A is *sequentially closed* in X if S is a sequence in A converging to a point $x \in X$, then $x \in A$. Obviously, every closed set is k-closed, and every k-closed set is sequentially closed in a space.

Definition 2.5. Let X be a topological space.

(1) The space X is a *sequential space* [11, p. 53] if every sequentially closed set of X is closed.

(2) The space X is a *Fréchet-Urysohn space* [11, p. 53] if, for each $A \subset X$ and $x \in \overline{A}$, there is a sequence in A converging to the point x in X.

(3) The space X is of *countable tightness* [33, Proposition 8.5] if, for any subset $A \subset X$ and $x \in \overline{A}$, there is a countable subset $C \subset A$ such that $x \in \overline{C}$.

(4) The space X is of *countable fan-tightness* [2, p. 565] if, for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets in X with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is a finite subset B_n of A_n for each $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$.

(5) The space X is of the *strong Pytkeev property* [45, Definition 5] if it has a countable (strict) Pytkeev network at each point $x \in X$.

(6) The space X is of *countable cn-character* [15, Definition 2.11] if it has a countable cn-network at each point $x \in X$.

The following are known.

Lemma 2.6. *(1) A topological space is first-countable if and only if it has the strong Pytkeev property and countable fan-tightness [6, Proposition 1.6].*

(2) Every space with countable cn-character has countable tightness [15, Proposition 2.3].

These notions in Definition 2.5 relate as follows [15]:

$$
\begin{tikzcd}
\text{First countability} \arrow[r] \arrow[dr] \arrow[dd] & \text{Fréchet-Urysohn space property} \arrow[r] & \text{sequential space property} \arrow[r] \arrow[dd] & k\text{-space property} \\
& \text{countable fan-tightness} \arrow[dr] & & \\
\text{strong Pytkeev property} \arrow[r] & \text{countable } cn\text{-character} \arrow[r] & \text{countable tightness} &
\end{tikzcd}
$$

Figure 3 The relationship among certain countability

3. Some Relationships Among Spaces with Certain Networks

In this section, we discuss Problem 1.1, obtain the relationship among spaces defined by Pytkeev networks, k-networks, wcs^*-networks, cs'-networks and cn-networks, and establish a metrizable theorem as follows: a space is metrizable if and only if it is an M-space with a point-countable Pytkeev network, see Corollary 3.4.

A family $\mathcal{P}$ of subsets of a space X is *point-countable* [8, p. 350] if the family $\{P \in \mathcal{P} : x \in P\}$ is countable for each $x \in X$. T. Banakh [4, Proposition 1.6] proved that each countable Pytkeev network is a k-network. The following is a stronger form.

Theorem 3.1. *Every point-countable Pytkeev network for a space is a quasi-k-network.*

Proof. Let X be a topological space, and $\mathcal{P}$ a point-countable Pytkeev network for X. For each open subset U of X and each countably compact subset K of U, let $\mathcal{P}_U = \{P \in \mathcal{P} : P \subset U\}$. Suppose that we have $K \setminus \bigcup \mathcal{F} \neq \emptyset$, for each finite subfamily $\mathcal{F}$ of $\mathcal{P}_U$. Clearly, for each $x \in K$, $\mathcal{P}_x = \{P \in \mathcal{P}_U : x \in P\}$ is countable, and let $\mathcal{P}_x = \{P_i(x)\}_{i\in\mathbb{N}}$. Take a point $x_1 \in K$. There exists a point $x_2 \in K$ such that $x_2 \in K \setminus P_1(x_1)$. By induction, we can choose a sequence $\{x_k\}_{k\in\mathbb{N}} \subset K$ such that $x_{k+1} \in K \setminus \bigcup_{i,n\leq k} P_i(x_n)$ for each $k \in \mathbb{N}$. Clearly, $x_k \notin P_i(x_n)$ for each $i, n < k$. Since K is a countably compact set, the sequence $\{x_k\}_{k\in\mathbb{N}}$ has an accumulation point $y \in K \subset U$. The point y is an accumulation point of the set $A = \{x_k : k \in \mathbb{N}\}$. Because $\mathcal{P}$ is a Pytkeev network for X, there exists a set $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap A$ is infinite. Take $i \in \mathbb{N}$ with $x_i \in P \subset U$. Thus we can choose $m \in \mathbb{N}$ such that $P = P_m(x_i)$. But $x_k \notin P_m(x_i)$ for each $k > m, i$. This shows that the set $P \cap A$ is finite, which is a contradiction. Thus $\mathcal{P}$ is a quasi-k-network for X. $\square$

Remark 3.2. The converse of Theorem 3.1 is not true. Let $X = \mathbb{N} \cup \{p\} \subset \beta\mathbb{N}$, where $\beta\mathbb{N}$ is the Čech-Stone compactification of $\mathbb{N}$ and $p \in \beta\mathbb{N} \setminus \mathbb{N}$. It is easy to check that each countably compact subset of X is finite. Thus the family $\{\{x\} : x \in X\}$ is a countable quasi-k-network for X. It was proved that X does not have a point-countable Pytkeev network [4, Example 1.11].

The point-countability in Theorem 3.1 can not be omitted.

Example 3.3. A strict Pytkeev network for a compact space X which is not a k-network.

Let $X = [0, \omega_1]$, and it be endowed with the ordered topology. Then X is a compact space. Let L be the set of all limit ordinals in X. For each $\alpha < \omega_1$ and each $n < \omega$, let $P_{\alpha,n} = (\{\beta + n : \beta \in L\} \cap (\alpha, \omega_1]) \cup \{\omega_1\}$. Clearly, $(\alpha, \omega_1] = \bigcup_{n<\omega} P_{\alpha,n}$. For each $x \in X$, define a family $\mathcal{P}_x$ of subsets of X as follows: if $x \neq \omega_1$, let $\mathcal{P}_x$ be a countable local base of x with $\bigcup \mathcal{P}_x \subset [0, x]$; if $x = \omega_1$, let $\mathcal{P}_x = \{P_{\alpha,n} : \alpha < \omega_1, n < \omega\}$. Put $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$. Obviously, $|\{P \in \mathcal{P} : \omega_1 \in P\}| = \omega_1$. Since X is a compact space and X is not covered by any finitely many elements of $\mathcal{P}$, $\mathcal{P}$ is not a k-network for X. We will prove that $\mathcal{P}$ is a strict Pytkeev network for X. Clearly, $\mathcal{P}$ is a network for X. For each $x \in X$, let U be a neighborhood of the point x in X, and A be a subset of X accumulating at x. Without loss of generality, we may assume that $x = \omega_1$, there is an ordinal $\alpha < \omega_1$ such that $(\alpha, \omega_1] \subset U$. Then $A \cap (\alpha, \omega_1]$ is an uncountable set because x is an accumulation point of the set A. By $A \cap (\alpha, \omega_1] = \bigcup_{n<\omega} A \cap P_{\alpha,n}$, there exists $n < \omega$ such that $A \cap P_{\alpha,n}$ is infinite. It is obvious that $x \in P_{\alpha,n} \subset U$. Therefore, $\mathcal{P}$ is a strict Pytkeev network for X.

A space X is an *M-space* [34, p. 150] if there are a metric space M and a closed mapping $f : M \to X$ such that each $f^{-1}(x)$ is countably compact. Obviously, every metric space and every countably compact space are an M-space. Since a space is metrizable if and only if it is an M-space with a point-countable quasi-k-network [21, Corollary 4.2], the following is obtained.

Corollary 3.4. *A space is metrizable if and only if it is an M-space with a point-countable Pytkeev network.*

Theorem 3.5. *Every wcs*-network in a sequential space is a Pytkeev network.*

Proof. Let X be a sequential space, and $\mathcal{P}$ a wcs^*-network for X. Take any subset $A \subset X$ accumulating at a point $x \in X$ and an open subset $O \subset X$ with $x \in O$. Let $B = (A \setminus \{x\}) \cup (X \setminus O)$. Then $x \in \overline{B} \setminus B$. Since X is a sequential space, the set B is not sequentially closed in X, so there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in B converging to a point $z \notin B$. We can assume each $x_n \in A$ because $X \setminus O$ is closed, and all x_n differ from each other. By $X \setminus O \subset B$, the point $z \in O$. Since $\mathcal{P}$ is a wcs^*-network for X, there exists a set $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset O$ for some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$. It shows that $P \cap A$ is infinite. Thus $\mathcal{P}$ is a Pytkeev network for X. $\square$

Remark 3.6. The Čech-Stone compactification $\beta\mathbb{N}$ is a compact space with a point-countable cs^*-network, and does not have a point-countable Pytkeev network. In fact, since $\beta\mathbb{N}$ has no any non-trivial convergent sequences [11, Corollary 3.6.15], the family $\{\{x\} : x \in \beta\mathbb{N}\}$ is a point-countable cs^*-network for $\beta\mathbb{N}$. By Corollary 3.4, $\beta\mathbb{N}$ does not have a point-countable Pytkeev network.

Lemma 3.7. *Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is a cn-network for X if and only if $\mathcal{P}$ is a network for X, and for each neighborhood U of a point x in X and each subset A of X accumulating at x, there exists a set $P \in \mathcal{P}$ such that $\{x, z\} \subset P \subset U$ for some $z \in A \setminus \{x\}$.*

Proof. Suppose $\mathcal{P}$ is a *cn*-network for X. Obviously, $\mathcal{P}$ is a network for X. For each neighborhood U of a point x in X and each subset A of X accumulating at x, since the set $\bigcup\{P \in \mathcal{P} : x \in P \subset U\}$ is a neighborhood of x, there is a point $z \in (A \setminus \{x\}) \cap \bigcup\{P \in \mathcal{P} : x \in P \subset U\}$, thus $\{x, z\} \subset P \subset U$ for some $P \in \mathcal{P}$, i.e., there exists a set $P \in \mathcal{P}$ such that $\{x, z\} \subset P \subset U$ for some $z \in A \setminus \{x\}$.

On the other hand, let U be a neighborhood of a point x in X, and put $V = \bigcup\{P \in \mathcal{P} : x \in P \subset U\}$. If the set V is not a neighborhood of x, then $x \in \overline{X \setminus V}$. Since $\mathcal{P}$ is a network for X, $x \in V$, thus x is an accumulation point of the set $X \setminus V$. By the condition of the sufficiency, there exists a set $P \in \mathcal{P}$ such that $\{x, z\} \subset P \subset U$ for some $z \in X \setminus V$, which is a contradiction. Therefore, $\mathcal{P}$ is a *cn*-network for X. □

Corollary 3.8. *Each cn-network for a space is a cs′-network.*

Theorem 3.9. *Each cs′-network in a Fréchet-Urysohn space is a cn-network.*

Proof. Let X be a Fréchet-Urysohn space, and $\mathcal{P}$ a *cs*′-network for X. For each $x \in X$ and each neighborhood U of x, let $V = \bigcup\{P \in \mathcal{P} : x \in P \subset U\}$. If $x \notin V^\circ$, then $x \in \overline{X \setminus V}$. Because X is a Fréchet-Urysohn space, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X \setminus V$ converging to the point x in X. Since $\mathcal{P}$ is a *cs*′-network for X, there exist $m \in \mathbb{N}$ and $P \in \mathcal{P}$ such that $\{x, x_m\} \subset P \subset U$. Thus $P \subset V$ and $P \cap (X \setminus V) \neq \emptyset$, which is a contradiction. Thus the set V is a neighborhood of the point x in X. Hence $\mathcal{P}$ is a *cn*-network for X. □

Remark 3.10. (1) There is a first-countable space X with a point-countable Pytkeev network, but without a point-countable *cs*′-network, see Example 4.8.

(2) There is a sequential space with a point-countable *cs**-network, but without a point-countable *cn*-network, see the space X in Example 5.13.

(3) There is a Fréchet-Urysohn space with a point-countable *cn*-network, but without a point-countable *wcs**-network.

Indeed, there exists a non-first-countable, Fréchet-Urysohn, countable and regular space S [40, Example 2.3]. It was showed that the space S does not have a point-countable k-network [26, Remark 2.10(4)]. By Theorems 3.1 and 3.5, the space S does not have a point-countable *wcs**-network. Write $S = \{s_n : n \in \mathbb{N}\}$. Let $\mathcal{P} = \{\{s_n, s_m\} : n, m \in \mathbb{N}\}$. Then $\mathcal{P}$ is a countable *cs*′-network for S. Namely, $\mathcal{P}$ has a countable *cn*-network for S by Theorem 3.9.

A topological space X is a *q-space* [31, p. 173] if, for each $x \in X$ there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X with $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point in X. Every M-space, thus every countably compact space, is a q-space. The following result improves A.V. Arhangel'skiĭ and A. Bella's theorem [2, Corollary 2]: a countably compact regular space has countable tightness if and only if it has countable fan-tightness.

Theorem 3.11. *Let X be a regular q-space. If X has countable tightness, then X has countable fan-tightness.*

Proof. Fix $x \in X$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Without loss of generality, we may assume that $x \notin A_n$ for every $n \in \mathbb{N}$. Because X is a q-space, there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of x which satisfies the condition of q-spaces. For each $n \in \mathbb{N}$, let $B_n = A_n \cap U_n$. Clearly, $x \in \bigcap_{n\in\mathbb{N}} \overline{B_n}$. Since X is a q-space, every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \in B_n$ for each $n \in \mathbb{N}$ has an accumulation point in X. Let $\mathcal{H} = \{\{x_n\}_{n\in\mathbb{N}} : x_n \in B_n \text{ for each } n \in \mathbb{N}\}$. For each $H \in \mathcal{H}$, let z_H be an accumulation point of the sequence H in X. Let $S = \{z_H : H \in \mathcal{H}\}$. We claim that $x \in \overline{S}$. Assume the contrary. Then the set $U = X \setminus \overline{S}$ is an open neighborhood of x. Since X is a regular space, there is an open set V in X such that $x \in V \subset \overline{V} \subset U$. Clearly, $V \cap B_n \neq \emptyset$ for each $n \in \omega$. Therefore we can choose $x_n \in V \cap B_n$ for each $n \in \mathbb{N}$. So there is the accumulation point z_C of the sequence $C = \{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Clearly, $z_C \in S$. But

$z_C \in \overline{V} \subset U = X \setminus \overline{S}$, which is a contradiction. Since X has countable tightness, there exists a countable set $M \subset S$ such that $x \in \overline{M}$. Put $M = \{y_n : n \in \mathbb{N}\}$, and for every $n \in \mathbb{N}$ select a sequence $D_n = \{z_{n,i}\}_{i\in\mathbb{N}} \in \mathcal{H}$ such that $y_n = z_{D_n}$ is the accumulation point of the sequence D_n. For every $n \in \mathbb{N}$, define $K_n = \{z_{1,n}, z_{2,n}, ..., z_{n,n}\}$ and $K = \bigcup_{n\in\mathbb{N}} K_n$. Clearly, $K_n \subset B_n \subset A_n$ for each $n \in \mathbb{N}$. Let W be an open neighborhood of x. We have $W \cap M \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $y_n \in W \cap M$. Because D_n accumulating at y_n, there exists $m > n$ such that $z_{n,m} \in W \cap K_m$. We conclude that $W \cap K \neq \emptyset$. It shows that $x \in \overline{K} = \overline{\bigcup_{n\in\mathbb{N}} K_n}$. Hence X has countable fan-tightness. □

The following is obtained by Lemma 2.6(1).

Corollary 3.12. *A regular q-space with the strong Pytkeev property is a first-countable space.*

Problem 3.13. *Is a q-space with a countable cn-network a first-countable space?*

4. Covering Properties on Spaces with Certain *cn*-Networks

In this section, we discuss covering properties on spaces with certain *cn*-networks and strict Pytkeev networks, prove that a topological space with a point-countable *cn*-network is a meta-Lindelöf D-space, which give an affirmative answer to Problem 1.2 and a partial answer to Problem 1.3, and obtain some characterizations of spaces with a locally countable strict Pytkeev network (*cn*-network).

A topological space X is called a *meta-Lindelöf space* [8, p. 370] if every open cover of X has a point-countable open refinement. A *neighborhood assignment* for a topological space (X, τ) is a function $\varphi : X \to \tau$ satisfying $x \in \varphi(x)$ for each $x \in X$. A space X is called a *D-space* [10] if, for each neighborhood assignment φ for X, there is a discrete closed set D in X such that $\{\varphi(d) : d \in D\}$ covers X. E.K. van Douwen and W. Pfeffer [10, p. 371] raised several interesting problems on D-spaces. Among them, the problem that whether every regular Lindelöf space is a D-space is still open. There are some close connections between generalized metric spaces and D-spaces [20].

A space X is called a *hereditarily meta-Lindelöf* (resp. *hereditary D-*) *space* if every subspace of X is a meta-Lindelöf (resp. D-) space.

Theorem 4.1. *Every space with a point-countable cn-network is a hereditarily meta-Lindelöf space.*

Proof. Suppose $\mathcal{P}$ is a point-countable *cn*-network for a topological space X. It is easy to verify that the property with a point-countable *cn*-network is hereditary with respect to subspaces, so we only need to prove that X is a meta-Lindelöf space.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ be an open cover of X, where γ is an ordinal. For every $\alpha < \gamma$, let

$$V_\alpha = \Big(\bigcup\{P \in \mathcal{P} : P \subset U_\alpha, P \not\subset U_\beta \text{ if } \beta < \alpha\}\Big)^\circ.$$

It is possible that some V_α are empty. Clearly, $V_\alpha \subset U_\alpha$. Next we will show that the family $\mathcal{V} = \{V_\alpha : \alpha < \gamma\}$ is a point-countable open cover of X. For each $x \in X$, let $\alpha(x) = \min\{\alpha < \gamma : x \in U_\alpha\}$, then $x \in U_{\alpha(x)}$. Since $\mathcal{P}$ is a *cn*-network for X, $\bigcup\{P \in \mathcal{P} : x \in P \subset U_{\alpha(x)}\}$ is a neighborhood of x. It is obvious that

$$\{P \in \mathcal{P} : x \in P \subset U_{\alpha(x)}\} \subset \{P \in \mathcal{P} : P \subset U_{\alpha(x)}, P \not\subset U_\beta \text{ if } \beta < \alpha(x)\}.$$

It shows that

$$\begin{aligned} x &\in \Big(\bigcup\{P \in \mathcal{P} : x \in P \subset U_{\alpha(x)}\}\Big)^\circ \\ &\subset \Big(\bigcup\{P \in \mathcal{P} : P \subset U_{\alpha(x)}, P \not\subset U_\beta \text{ if } \beta < \alpha(x)\}\Big)^\circ \\ &= V_{\alpha(x)}. \end{aligned}$$

So $\mathcal{V}$ is an open cover of X.

We claim that $\mathcal{V}$ is point-countable. Assuming the contrary, there exist a point $x \in X$ and an uncountable subset Γ of γ such that $x \in V_\alpha$ for each $\alpha \in \Gamma$. Let $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$. For each $\alpha \in \Gamma$, by $x \in V_\alpha$, we can choose a set $P_\alpha \in \mathcal{P}_x$ such that $P_\alpha \subset U_\alpha$ and $P_\alpha \not\subset U_\beta$ for each $\beta < \alpha$. Because $\mathcal{P}_x$ is a countable family and Γ is an uncountable set, we can assume that $P_\alpha = P_\beta$ for each $\alpha, \beta \in \Gamma$. Fixed different points $\alpha, \beta \in \Gamma$ with $\beta < \alpha$, then $U_\beta \supset P_\beta = P_\alpha \not\subset U_\beta$, which is a contradiction. So $\mathcal{V}$ is a point-countable family in X. It shows that X is a hereditarily meta-Lindelöf space. □

By Theorem 3.9 we have the following corollary, which gives an affirmative answer to Problem 1.2.

Corollary 4.2. *Every Fréchet-Urysohn space with a point-countable cs′-network is a hereditarily meta-Lindelöf space.*

Corollary 4.3. *Every space with a point-countable strict Pytkeev network is a hereditarily meta-Lindelöf space.*

Corollary 4.4. *Every space with a point-countable cn-network consisting of separable subsets is a topological sum of separable and Lindelöf spaces.*

Proof. Let $\mathcal{P}$ be a point-countable *cn*-network consisting of separable subsets for a topological space X. By Theorem 4.1, X is a meta-Lindelöf space. X is also a locally separable space, because for each $x \in X$, the set $\bigcup\{P \in \mathcal{P} : x \in P\}$ is a separable neighborhood of x. By [21, Proposition 8.7], X is a topological sum of Lindelöf spaces. It is easy to see that every Lindelöf locally separable space is separable. Hence X is a topological sum of separable and Lindelöf spaces. □

Let $\mathcal{P}$ be a family of subsets of a topological space X. The family $\mathcal{P}$ is said to be *star-countable* [8, p. 368] if $\{P \in \mathcal{P} : P \cap Q \neq \emptyset\}$ is countable for each $Q \in \mathcal{P}$. The family $\mathcal{P}$ is said to be *locally countable* [8, p. 349] if, for each $x \in X$, there is a neighborhood V of x such that V meets at most countably many elements of $\mathcal{P}$. The union of countably many locally countable families is called *σ-locally countable*.

Theorem 4.5. *The following are equivalent for a topological space X.*

(1) X has a locally countable strict Pytkeev network.

(2) X has a star-countable strict Pytkeev network.

(3) X has a σ-locally countable strict Pytkeev network consisting of separable subsets.

(4) X is a topological sum of spaces with a countable (strict) Pytkeev network.

Proof. (1) ⇒ (2) Let $\mathcal{P}$ be a locally countable strict Pytkeev network for a topological space X. For each $x \in X$, there exists a neighborhood V_x of x such that V_x meets at most countably many elements of $\mathcal{P}$. Let $\mathcal{P}^* = \{P \in \mathcal{P} : P \subset V_x \text{ for some } x \in X\}$. It is easy to verify that $\mathcal{P}^*$ is a star-countable strict Pytkeev network for X.

(2) ⇒ (3) Let $\mathcal{P}$ be a star-countable strict Pytkeev network for X. For each $P \in \mathcal{P}$, the family $\mathcal{P}|_P = \{Q \cap P : Q \in \mathcal{P}\}$ is a countable network for P, thus P is a separable subspace of X. For each $x \in X$, let $U = \bigcup\{P \in \mathcal{P} : x \in P\}$. Then U is a neighborhood of x. Because $\mathcal{P}$ is a star-countable family, U meets at most countably many elements of $\mathcal{P}$. Thus $\mathcal{P}$ is a locally countable family. So X has a σ-locally countable strict Pytkeev network consisting of separable subsets.

(3) ⇒ (4) Let $\mathcal{P}$ be a σ-locally countable strict Pytkeev network consisting of separable subsets for X. For each $P \in \mathcal{P}$, by Corollary 4.3, P is a meta-Lindelöf space, thus P is a Lindelöf space. Because every locally countable family in a Lindelöf space is countable, $\mathcal{P}$ is a star-countable family in X. Thus $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha \in \Lambda\}$, where each $\mathcal{P}_\alpha$ is countable and $(\bigcup \mathcal{P}_\alpha) \cap (\bigcup \mathcal{P}_\beta) \neq \emptyset$ if and only if $\mathcal{P}_\alpha = \mathcal{P}_\beta$ [8, Lemma 3.10]. For each $\alpha \in \Lambda$, let $X_\alpha = \bigcup \mathcal{P}_\alpha$. If $x \in X_\alpha$, then $\bigcup\{P \in \mathcal{P} : x \in P\} \subset X_\alpha$. Thus X_α is a neighborhood of x. It shows that X_α is an open set in X. So we have that $X = \bigoplus_{\alpha \in \Lambda} X_\alpha$. Clearly, every $\mathcal{P}_\alpha$ is a countable strict Pytkeev network for X_α. Thus X is a topological sum of spaces with a countable strict Pytkeev network.

(4) ⇒ (1) It is well known that each space with a countable Pytkeev network has a countable strict Pytkeev network [4, p. 152]. Thus, if X is a topological sum of spaces with a countable Pytkeev network, then X has a locally countable strict Pytkeev network. □

Remark 4.6. By the proof of Theorem 4.5, the results are also true if the strict Pytkeev networks are replaced by the *cn*-networks in Theorem 4.5.

Corollary 4.7. *Each regular space satisfying one of the following conditions is a topological sum of $\mathfrak{P}_0$-spaces, thus it is a paracompact $\mathfrak{P}$-space.*
(1) A space with a locally countable strict Pytkeev network.
(2) A k-space with a locally countable k-network.

Proof. If a regular space with a locally countable strict Pytkeev network, it is a topological sum of $\mathfrak{P}_0$-spaces by Theorem 4.5. Next, suppose a regular k-space X with a locally countable k-network. By [28, Corollary 2.8.11], X is a topological sum of sequential spaces with a countable cs^*-network. By Theorem 3.5, X is a topological sum of spaces with a countable Pytkeev network, i.e., X is a topological sum of $\mathfrak{P}_0$-spaces. □

Example 4.8. There is a first-countable space X with a star-countable, locally countable and σ-discrete Pytkeev network consisting of separable subsets, but without a point-countable cs'-network.

Let $S = \{(x, y) : x, y \in \mathbb{R}, y > 0\}$, $L = \{(x, 0) : x \in \mathbb{R}\}$ and $X = S \cup L$. Denote the Euclidean subspace topology on X by τ^*. Define the *half-disc topology* on X as follows [43, p. 96]:

$$\tau = \left\{\tau^*\right\} \bigcup \left\{\{x\} \cup (S \cap U) : x \in L, x \in U \in \tau^*\right\}.$$

Then (X, τ) is called a *half-disc topological space*. Let X be the half-disc topological space. It is easy to verify that X is a separable first-countable and non-regular space. But X is not a Lindelöf space. It shows that X is not a meta-Lindelöf space. By Corollary 4.2, X does not have a point-countable cs'-network.

For every $x \in \mathbb{R}^2, r > 0$, let $B(x, r)$ be a spherical neighborhood of the point x with a radius r in $\mathbb{R}^2$. Define

$$\mathcal{P} = \left\{\{p\} : p \in L\right\} \bigcup \left\{B(q, 1/n) \cap S : q \in \mathbb{Q} \times \mathbb{Q}, n \in \mathbb{N}\right\}.$$

Since L is a discrete closed set in X, $\mathcal{P}$ is a star-countable, locally countable and σ-discrete family consisting of separable subsets for X. We prove $\mathcal{P}$ is a wcs^*-network for X. Assume that $p \in U \in \tau$ and a sequence $\{p_k\}_{k\in\mathbb{N}}$ in U converges to the point p in X. Since L is discrete, we may assume all $p_k \in S$, and hence $\{p_k\}_{k\in\mathbb{N}}$ also converges to p in the Euclidean subspace topology τ^* on X. Since $\{B(q, 1/n) \cap X : q \in \mathbb{Q} \times \mathbb{Q}, n \in \mathbb{N}\}$ is a countable base of τ^*, there exist $q \in \mathbb{Q} \times \mathbb{Q}$ and $i, m \in \mathbb{N}$ such that $\{p\} \cup \{p_k : k \geq i\} \subset B(q, 1/m) \cap X \subset U$, and hence $\{p_k : k \geq i\} \subset B(q, 1/m) \cap S \subset U$. Thus, $\mathcal{P}$ is a wcs^*-network for X. By Theorem 3.5, $\mathcal{P}$ is a Pytkeev network for X.

By the Bing-Nagata-Smirnov metrization theorem, we have the following problem.

Problem 4.9. *Do regular spaces with a σ-locally finite (strict) Pytkeev network have a σ-discrete (strict) Pytkeev network?*

The following interesting result is a corollary of Peng [37, Corollary 18].

Theorem 4.10. *Every space with a point-countable cn-network is a hereditary D-space.*

Proof. We only need to prove X is a D-space. Let $\mathcal{P}$ be a point-countable cn-network for a topological space X. By Lemma 3.7, the following is satisfied: for each neighborhood U of a point x in X and each subset A of X with $x \in \overline{A} \setminus A$, there exists a set $P \in \mathcal{P}$ with $x \in P \subset U$ and $P \cap A \neq \emptyset$. It was proved by Peng [37, Corollary 18], X is a D-space. □

Since each countably compact D-space is compact [7, Proposition 1.4], the following is obvious.

Corollary 4.11. *Every countably compact space with a point-countable cn-network is a compact space.*

Corollary 4.12. *Every space with a point-countable strict Pytkeev network is a hereditary D-space.*

By Theorem 3.5, Problem 1.3 is equivalent to whether every sequential space with a point-countable Pytkeev network is a D-space. Corollary 4.12 gives a partial answer to Problem 1.3.

Problem 4.13. *Do regular spaces with a σ-locally finite cn-network have a σ-discrete cn-network?*

It is known that every monotonically normal D-space is paracompact [7, Theorem 17]. It follows from Theorem 4.10 that every monotonically normal space with a point-countable *cn*-network is paracompact. Moreover, every normal k-and $\aleph$-space is paracompact [13, Theorem 1.3]. The following problem is posed.

Problem 4.14. *Are normal $\mathfrak{P}$-spaces paracompact?*

5. Mapping Properties on Spaces with Pytkeev Networks

In this section, we prove that (strict) Pytkeev networks are preserved by closed mappings and finite-to-one pseudo-open mappings, and *cn*-networks are preserved by pseudo-open mappings, in particular, spaces with a point-countable Pytkeev network are preserved by closed mappings. Some examples showing spaces with certain Pytkeev networks are not be preserved by some mappings are constructed.

Recall some concepts relevant to mappings. Let $f : X \to Y$ be a mapping. f is called a *quotient mapping* [11, p. 91] if, for each $U \subset Y$ with $f^{-1}(U)$ open in X, U is open in Y. f is called a *pseudo-open mapping* [1] if, for each $y \in Y$ and $f^{-1}(y) \subset V$ with V open in X, $y \in [f(V)]^{\circ}$. f is called a *countable-to-one* (resp. *finite-to-one*, *compact*) *mapping* if every $f^{-1}(y)$ is a countable (resp. finite, compact) subset of X. f is called a *perfect mapping* [11, p. 182] if, f is a closed and compact mapping.

Obviously, every closed mapping or open mapping is a pseudo-open mapping, and every pseudo-open mapping is a quotient mapping.

Theorem 5.1. *Each (strict) Pytkeev network is preserved by a closed mapping.*

Proof. Let $\mathcal{P}$ be a (strict) Pytkeev network for a topological space X and $f : X \to Y$ be a closed mapping. Take any set $A \subset Y$ accumulating at a point $y \in Y$. Given an open neighborhood $O \subset Y$ of y, we need to find a set $P \in \mathcal{P}$ such that $(y \in) f(P) \subset O$ and $f(P) \cap A$ is infinite. For each $z \in A \setminus \{y\}$, fix a point $x_z \in f^{-1}(z)$. Let $B = \{x_z : z \in A \setminus \{y\}\}$. Then $f(B) = A \setminus \{y\}$. Since f is a closed mapping, $y \in \overline{A \setminus \{y\}} = f(\overline{B})$. So we can choose a point $x \in f^{-1}(y) \cap \overline{B}$, then $f(x) = y$ and $x \notin B$. Thus x is an accumulation point of the set B. Since f is continuous and $\mathcal{P}$ is a (strict) Pytkeev network for X, there exists a set $P \in \mathcal{P}$ such that $(x \in) P \subset f^{-1}(O)$ and $P \cap B$ is infinite. Thus $(y \in) f(P) \subset O$ and $f(P \cap B) \subset f(P) \cap A$ is infinite. It shows that the family $\{f(P) : P \in \mathcal{P}\}$ is a (strict) Pytkeev network for Y. □

Corollary 5.2. *Spaces with a σ-locally finite (strict) Pytkeev network are preserved by perfect mappings. Consequently, $\mathfrak{P}$-spaces are preserved by perfect mappings.*

Proof. Let $f : X \to Y$ be a perfect mapping, and the family $\mathcal{P}$ be a σ-locally finite (strict) Pytkeev network for the space X. By Theorem 5.1, the family $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ is a (strict) Pytkeev network for the space Y. Since each locally finite family is preserved by a perfect mapping [11, Lemma 3.10.11], $f(\mathcal{P})$ is a σ-locally finite family of Y. Hence, the space Y has a σ-locally finite (strict) Pytkeev network. □

Corollary 5.3. *Each point-countable strict Pytkeev network is preserved by a countable-to-one closed mapping.*

Proof. Let $f : X \to Y$ be a countable-to-one closed mapping, and $\mathcal{P}$ be a point-countable strict Pytkeev network for the space X. By Theorem 5.1, the family $\{f(P) : P \in \mathcal{P}\}$ is a strict Pytkeev network for Y. Because f is a countable-to-one mapping, $\{f(P) : P \in \mathcal{P}\}$ is a point-countable family in Y. Thus Y has a point-countable strict Pytkeev network. □

The following example shows that the condition "each $f^{-1}(y)$ is countable" in Corollary 5.3 can not be replaced by "each $f^{-1}(y)$ is Lindelöf".

Example 5.4. Spaces with a point-countable strict Pytkeev network do not be preserved by closed mappings with Lindelöf fibers.

Let $\mathbb{I}$ be the closed unit interval $[0, 1]$ with the usual topology, and B be a Bernstein set in $\mathbb{I}$ [11, p. 339]. We denote by $\mathbb{I}_B$ the space obtained from $\mathbb{I}$ by isolating the points of B. Let $\mathbb{S}_1 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual topology. Put

$$X = \mathbb{I}_B \times \mathbb{S}_1, \quad A = \mathbb{I}_B \times \{0\} \quad \text{and} \quad Y = X/A.$$

Then A is a closed subspace of X and the quotient mapping $q : X \to Y$ is a closed mapping. It was proved that the space X is a regular Lindelöf space with a point-countable base and the space Y does not have any point-countable cs^*-network [42, Theorem 2.4]. Obviously, X has a point-countable strict Pytkeev network, and Y does not have any point-countable strict Pytkeev network.

It is known that spaces with a point-countable k-network do not be preserved by closed mappings [41, Example 1], and spaces with a point-countable cs^*-network do not be preserved by perfect mappings [28, Example 3.1.20(8)].

Theorem 5.5. *Spaces with a point-countable Pytkeev network are preserved by closed mappings.*

Proof. Let $\mathcal{P}$ be a point-countable Pytkeev network for a topological space X and $f : X \to Y$ be a closed mapping. For each $y \in Y$, take a point $x_y \in f^{-1}(y)$. Put $Q = \{x_y : y \in Y\}$. Let $\mathcal{Q} = \{f(P \cap Q) : P \in \mathcal{P}\}$. If a point $y \in f(P \cap Q)$ for some $P \in \mathcal{P}$, then $f^{-1}(y) \cap P \cap Q \neq \emptyset$, i.e., $x_y \in P$. Since $\mathcal{P}$ is point-countable in X, the family $\mathcal{Q}$ is also point-countable in Y. Next, we will show that $\mathcal{Q}$ is a Pytkeev network for Y.

Let A be a subset of Y accumulating at a point $y \in Y$ and O an open subset with $y \in O \subset Y$. Let $B = \{x_z : z \in A \setminus \{y\}\}$. By the proof of Theorem 5.1, there are a point $x \in f^{-1}(y) \cap (\overline{B} \setminus B)$ and a neighborhood V of x such that $f(V) \subset O$. Since $\mathcal{P}$ is a Pytkeev network for X, there exists a set $P \in \mathcal{P}$ such that $P \subset V$ and $P \cap B$ is infinite. Thus $f(P \cap Q) \subset f(P) \subset f(V) \subset O$ and $f(P \cap B) \cap A \subset f(P \cap Q) \cap A$ is infinite. It shows that $\mathcal{Q}$ is a Pytkeev network for Y. □

Corollary 5.6. ([27, Theorem 5]) *k-spaces with a point-countable (quasi-)k-network are preserved by closed mappings.*

Proof. Let $f : X \to Y$ be a closed mapping and the space X be a k-space with a point-countable (quasi-)k-network. Since k-spaces are preserved by quotient mappings [11, Theorem 3.3.23], the space Y is a k-space. By Lemma 2.3 and Theorem 5.5, the space Y has a point-countable Pytkeev network. It follows from Theorem 3.1 that Y has a point-countable (quasi-)k-network. □

Theorem 5.7. *Each (strict) Pytkeev network is preserved by a finite-to-one pseudo-open mapping.*

Proof. Let $\mathcal{P}$ be a (strict) Pytkeev network for a topological space X and $f : X \to Y$ be a finite-to-one pseudo-open mapping. Take any set $A \subset Y$ accumulating at a point $y \in Y$, i.e., $y \in \overline{A \setminus \{y\}}$. Then we have $f^{-1}(y) \bigcap \overline{f^{-1}(A \setminus \{y\})} \neq \emptyset$. Otherwise, we have $f^{-1}(y) \subset X \setminus \overline{f^{-1}(A \setminus \{y\})}$. Because f is a pseudo-open mapping,

$$y \in [f(X \setminus \overline{f^{-1}(A \setminus \{y\})})]^{\circ} \subset Y \setminus \overline{A \setminus \{y\}},$$

which is a contradiction. So there exists a point $x \in f^{-1}(y) \cap \overline{f^{-1}(A \setminus \{y\})}$. Given an open neighborhood $O \subset Y$ of y, there exists a neighborhood V of x such that $f(V) \subset O$. Since $\mathcal{P}$ is a (strict) Pytkeev network for X, there exists a set $P \in \mathcal{P}$ such that $(x \in)P \subset V$ and $P \cap f^{-1}(A \setminus \{y\})$ is infinite. Since f is a finite-to-one mapping, $f(P) \cap A$ is infinite and $(y \in)f(P) \subset O$. It shows that the family $\{f(P) : P \in \mathcal{P}\}$ is a (strict) Pytkeev network for Y. □

Corollary 5.8. *Spaces with a point-countable (strict) Pytkeev network are preserved by finite-to-one pseudo-open mappings.*

Theorem 5.9. *Each cn-network is preserved by a pseudo-open mapping.*

Proof. Let $f : X \to Y$ be a pseudo-open mapping and the space X have a cn-network $\mathcal{P}$. We prove that the family $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ is a cn-network for the space Y. For each $y \in Y$ and each neighborhood U of y, let $W = \bigcup\{f(P) \in f(\mathcal{P}) : y \in f(P) \subset U\}$. For each $x \in f^{-1}(y)$, let $V_x = \bigcup\{P \in \mathcal{P} : x \in P \subset f^{-1}(U)\}$. Clearly, $f(V_x) \subset W$, and $x \in [V_x]^\circ$ because $\mathcal{P}$ is a cn-network for X. Then $f^{-1}(y) \subset \bigcup\{[V_x]^\circ : x \in f^{-1}(y)\}$. Since f is a pseudo-open mapping,

$$y \in \Big[f\Big(\bigcup\Big\{[V_x]^\circ : x \in f^{-1}(y)\Big\}\Big)\Big]^\circ \subset \bigcup\Big\{f(V_x) : x \in f^{-1}(y)\Big\} \subset W.$$

So W is a neighborhood of y. It shows that $f(\mathcal{P})$ is a cn-network for Y. □

Corollary 5.10. *Spaces with a point-countable cn-network are preserved by countable-to-one pseudo-open mappings.*

By Theorem 5.9 and the proof of Corollary 5.2, we have the following corollary.

Corollary 5.11. *Spaces with a σ-locally finite cn-network are preserved by perfect mappings. Consequently, strict σ-spaces are preserved by perfect mappings.*

Example 5.12. Spaces with a point-countable cn-network (cs'-network) are not preserved by closed mappings.

Let S_{ω_1} be the quotient space obtained by identifying all the limit points of the topological sum of ω_1 many non-trivial convergent sequences. Obviously, S_{ω_1} is a closed image of a metrizable space. It follows from [26, Remark 2.10(3)] that S_{ω_1} has not any point-countable cs'-network. Thus, S_{ω_1} has not any point-countable cn-network. Since each metrizable space has a point-countable cn-network, spaces with a point-countable cn-network (cs'-network) are not preserved by closed mappings.

Let $\mathcal{P}$ be a cover of a topological space X. The X is *determined by* the cover $\mathcal{P}$ (or has a *weak topology* with respect to $\mathcal{P}$) [21, p. 303], if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is relatively open (relatively closed) in P for each $P \in \mathcal{P}$. It is known that if the family $\mathcal{P}$ is a cover of X, then X is determined by $\mathcal{P}$ if and only if the obvious mapping $f : \bigoplus \mathcal{P} \to X$ is quotient, where $\bigoplus$ denotes topological sum [21, Lemma 1.8].

Example 5.13. Spaces with a point-countable strict Pytkeev network (cn-network) are not preserved by finite-to-one quotient mappings.

Let

$$X = \mathbb{I} \times \mathbb{S}_1,\ Y = \mathbb{I} \times (\mathbb{S}_1 \setminus \{0\}).$$

Define a topology for X as follows [21, Example 9.3]: Y is a Euclidean subspace of X and each element of a neighborhood base of a point $(t, 0) \in X$ has the form

$$\{(t, 0)\} \cup \Big(\bigcup\{V(t, k) : k \geq n\}\Big),\ \ n \in \mathbb{N},$$

where $V(t, k)$ is an open neighborhood of the point $(t, 1/k)$ in the subspace $\mathbb{I} \times \{1/k\}$.

Let

$$M = \Big(\bigoplus\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\}\Big) \oplus \Big(\bigoplus\{\{t\} \times \mathbb{S}_1 : t \in \mathbb{I}\}\Big).$$

Then M is a locally compact metrizable space, thus M has a point-countable strict Pytkeev network (cn-network). Let $f : M \to X$ be the obvious mapping. Since X is determined by the point-finite cover $\{\mathbb{I} \times \{1/n\} : n \in \mathbb{N}\} \bigcup \{\{t\} \times \mathbb{S}_1 : t \in \mathbb{I}\}$, f is a finite-to-one quotient mapping [21, Lemma 1.8].

Obviously, X is a separable regular space. Since $\mathbb{I} \times \{0\}$ is an uncountable discrete closed subspace of X, X is not a Lindelöf space. And since every separable meta-Lindelöf space is Lindelöf, X is not a meta-Lindelöf space. By Corollary 4.3 (Theorem 4.1), X does not have a point-countable strict Pytkeev network (cn-network).

The space X is a sequential space with a point-countable cs^*-network, because it is a finite-to-one quotient image of a metrizable space [44, Lemma 2.2].

Example 5.14. Spaces with a point-countable Pytkeev network are not preserved by countable-to-one open mappings.

This example was constructed in [46, Example 1], which shows that the Pytkeev property is not preserved by a countable-to-one open mapping. Since it was written in Chinese, we give the construction and proof of the example.

Let $X = \{0\} \cup \mathbb{N}^2$. Denote the family of all functions from $\mathbb{N}$ into $\mathbb{N}$ by $\mathbb{N}^{\mathbb{N}}$. Put $V(n,m) = \{(n,k) \in \mathbb{N}^2 : k \geq m\}$ for each $n, m \in \mathbb{N}$. For every $F \subset \mathbb{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$, let $H(F, f) = \bigcup\{V(n, f(n)) : n \in F\}$. Let p be a free ultrafilter on $\mathbb{N}$. Define a topology of X as follows: if $x \in X \setminus \{0\}$, $\{x\}$ is an open set; if $x = 0$, $\mathcal{U}_x = \{\{0\} \cup H(F, f) : F \in p, f \in \mathbb{N}^{\mathbb{N}}\}$ as a neighborhood base of x. Let $\mathcal{P} = \{V(n,m) : n, m \in \mathbb{N}\}$. Clearly, $\mathcal{P}$ is a countable Pytkeev network at the point 0. Thus X has a countable Pytkeev network.

Let $Y = \mathbb{N} \cup \{p\}$ where $p \in \beta\mathbb{N} \setminus \mathbb{N}$, and it be endowed with the subspace topology of $\beta\mathbb{N}$. Since the space Y does not have a countable Pytkeev network at the point p [4, Example 1.11], Y does not have the strong Pytkeev property, thus Y does not have a countable Pytkeev network. Define a mapping $f : X \to Y$ as follows: $f(0) = p$ and $f(\{(n,k) : k \in \mathbb{N}\}) = \{n\}$ for each $n \in \mathbb{N}$. Clearly, f is a countable-to-one mapping. We show that f is an open mapping. For each $U \in \tau(X)$, if $0 \notin U$, then $f(U) \subset \mathbb{N}$, since $\mathbb{N}$ is discrete, $f(U) \in \tau(Y)$; if $0 \in U$, then there exist $F = \{n_i : i \in \mathbb{N}\} \in p$ and $f \in \mathbb{N}^{\mathbb{N}}$ such that $\{0\} \cup H(F, f) \subset U$, so $\{p\} \cup F \subset f(U)$, we have $f(U) \in \tau(Y)$. Then f is an open mapping.

Problem 5.15. *Are spaces with a point-countable strict Pytkeev network preserved by perfect mappings.*

Acknowledgements

The authors would like to thank the referees for some constructive suggestions and all their efforts in order to improve this paper.

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