Topology and its Applications 235 (2018) 92–103

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A special class of semi(quasi)topological groups and three-space properties ☆

Zhongbao Tang [a,*], Shou Lin [b], Fucai Lin [c]

[a] School of Mathematics, Sichuan University, Chengdu 610065, PR China
[b] Institute of Mathematics, Ningde Normal University, Ningde 352100, PR China
[c] School of mathematics and statistics, Minnan Normal University, Zhangzhou 363000, PR China

ARTICLE INFO

Article history:
Received 2 June 2017
Received in revised form 4 December 2017
Accepted 4 December 2017
Available online 8 December 2017

MSC:
22A05
54A25
54E25
54E30
54E35
54G20
54H11

Keywords:
Moore space
Quasitopological group
Semitopological group
Sequentially continuous
Three-space property

ABSTRACT

The multiplication of a semitopological (quasitopological) group G is called sequentially continuous if the product map of $G \times G$ into G is sequentially continuous. In this paper, we mainly consider the properties of semitopological (quasitopological) groups with sequentially continuous multiplications and three-space problems in quasitopological groups. It is showed that (1) every snf-countable semitopological group G with the sequentially continuous multiplication is sof-countable; (2) if G is a sequential quasitopological group with the sequentially continuous multiplication, then G contains a closed copy of S_ω if and only if it contains a closed copy of S_2, which give a partial answer to a problem posed by R.-X. Shen; (3) let G be a quasitopological group with the sequentially continuous multiplication, then the following are equivalent: (i) G is a sequential α_4-space; (ii) G is Fréchet; (iii) G is strongly Fréchet; (4) (MA+¬CH) there exists a non-metrizable, separable, normal and Moore quasitopological group; (5) some examples are constructed to show that metrizability, first-countability and second-countability are not three-space properties in the class of quasitopological groups.

1. Introduction

Recall that a *paratopological group* G is a group endowed with a topology such that the multiplication of G is jointly continuous. A *semitopological group* G is a group endowed with a topology such that the multiplication of G is separately continuous. A *topological group* (resp., *quasitopological group*) is a paratopological group (resp., semitopological group) G such that the inversion of G is continuous.

☆ Supported by the NSFC (Nos. 11471153, 11571158).

* Corresponding author.
E-mail addresses: tzbao84@163.com (Z. Tang), shoulin60@163.com (S. Lin), linfucai2008@aliyun.com (F. Lin).

https://doi.org/10.1016/j.topol.2017.12.016
0166-8641/

As a generalization of topological groups, it is natural to consider which results valid for topological groups can be extended to semitopological groups or quasitopological groups [6]. Unfortunately, since the multiplication of a quasitopological group need not be continuous, some famous theorems of topological groups are not valid in quasitopological groups. For example, Comfort and Ross [11] proved that the product of an arbitrary family of pseudocompact topological groups is pseudocompact. However, C. Hernández and M. Tkachenko [21] constructed two pseudocompact quasitopological groups whose product fails to be pseudocompact. So it is suitable to consider a subclass of semitopological (quasitopological) groups.

The multiplication of a semitopological (quasitopological) group G is called *sequentially continuous* if the product map of $G \times G$ into G is sequentially continuous. It is equivalent to the condition that $a_n b_n \to e$ whenever $a_n \to e$ and $b_n \to e$, where e is the neutral element of the group G. Since the multiplication of a paratopological group G is continuous, it is sequentially continuous. Therefore, this subclass of semitopological (resp., quasitopological) groups contains paratopological (resp., topological) groups. In Section 3, we mainly consider semitopological groups with sequentially continuous multiplications. Some properties of this subclass of semitopological groups are obtained. We prove that every snf-countable semitopological group with the sequentially continuous multiplication is sof-countable. We also obtain some corollaries of this result.

In Section 4, the properties of quasitopological groups with sequentially continuous multiplications are discussed. It was proved in [31] that a topological group contains a (closed) copy of S_ω if and only if it contains a (closed) copy of S_2. R.-X. Shen pointed that there is a quasitopological group [35, Example 3.9] containing a closed copy of S_2. However, the quasitopological group contains no closed copy of S_ω. Therefore, the following problem was posed.

Problem 1.1. [35, Problem 3.11] Let G be a paratopological (quasitopological) group containing a closed copy of S_ω. Must G contain a closed copy of S_2?

We prove that if G is a sequential quasitopological group with the sequentially continuous multiplication, then G contains a closed copy of S_ω if and only if it contains a closed copy of S_2, which give a partial answer to Problem 1.1 for quasitopological groups. We also prove that every Fréchet quasitopological group with the sequentially continuous multiplication is strongly Fréchet. These results improve some relevant results in topological groups. At the end of this section, we construct under MA+¬CH a non-metrizable, separable, normal and Moore quasitopological group.

Let $\mathscr{P}$ be a (topological, algebraic, or mixed nature) property. We say that $\mathscr{P}$ is a *three-space property* if whenever a closed invariant subgroup N of a topological group G and the quotient group G/N have $\mathscr{P}$, so does G. Similarly one defines a three-space property in paratopological or quasitopological or semitopological groups. Three-space problems in topological groups are considered by many authors. The list of three-space properties in topological groups is quite long, it includes compactness, local compactness, pseudocompactness, precompactness, metrizability (first-countability), second-countability, connectedness, completeness, etc (see [6,9,10,12,27,38]). There are several papers which contain some results related to the three-space problem in the class of paratopological groups or semitopological groups (see [15,33,34,39]). Much less is known about three-space properties in quasitopological groups. Connectedness [39] and separability [15] are three-space properties in semitopological groups. It was pointed in [39] that compactness and local compactness are not three-space properties in quasitopological groups. M. Fernández and I. Sánchez showed that being a topological group is not a three-space property in the class of quasitopological groups [15, Example 2.9].

In Section 5, we continue the study of the three-space properties in the class of quasitopological groups. Some examples are constructed to show that metrizability, first-countability and second-countability are not three-space properties in the class of quasitopological groups.

2. Preliminaries

Let X be a space. For every $P \subseteq X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. The set P is a *sequentially open* subset of X if P is a sequential neighborhood of each point in P. The set P is a *sequentially closed* subset of X if $X \setminus P$ is sequentially open. A space X is said to be a *sequential space* [16] if each sequentially open subset is open in X. For each space (X, τ) the *sequential coreflection* [17] of (X, τ), denoted (X, σ_τ) or σX, is given by $U \in \sigma_\tau$ if and only if U is sequentially open in (X, τ). As it is well known, σX is a sequential space [17, p. 52]; also, X and σX have the same convergent sequences [8, p. 678]. A topological space X is called a *Fréchet space* [16] if for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X. Every Fréchet space is sequential.

Let X be a space and $x \in X$. Suppose that $\mathscr{P}$ is a family of subsets in X. The family $\mathscr{P}$ is called a *network* at x if $x \in \bigcap \mathscr{P}$ and for every open neighborhood U of x in X, there exists $P \in \mathscr{P}$ such that $P \subseteq U$.

Definition 2.1. Let $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) $\mathscr{P}_x$ is a network at x in X; (b) if $U, V \in \mathscr{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathscr{P}_x$.

(1) The family $\mathscr{P}$ is called an *sn-network* [25] for X if each element of $\mathscr{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. X is called *snf-countable* [25] if X has an *sn*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *sn-metrizable* [18] if X has a σ-locally finite *sn*-network.

(2) The family $\mathscr{P}$ is called an *so-network* [25] for X if each element of $\mathscr{P}_x$ is a sequentially open in X for each $x \in X$. X is called *sof-countable* [25] if X has an *so*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *so-metrizable* [18] if X has a σ-locally finite *so*-network.

(3) The family $\mathscr{P}$ is called a *weak base* [2] for X if for every $A \subseteq X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathscr{P}_x$ such that $P \subseteq A$. X is called *weakly first-countable* [2] if X has a weak base $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable.

A space X is weakly first-countable if and only if X is sequential and *snf*-countable [25,37].

All spaces are Hausdorff unless stated otherwise. We denote by $\mathbb{N}, \mathbb{Q}, \mathbb{R}$, ω and $\mathfrak{c}$ the set of all positive integers, the set of all rational numbers, the set of all real numbers, the first infinite ordinal and the cardinality of the continuum, respectively. The neutral element of a group is denoted by e. Readers may consult [6,14,19] for notation and terminology not given here.

3. Some properties of semitopological groups with sequentially continuous multiplications

In this section, we discuss some properties of semitopological groups with sequentially continuous multiplications.

Let X be an *snf*-countable space. Then it is easy to see that X has an *sn*-network $\{V_n(x) : x \in X, n \in \mathbb{N}\}$ such that the following conditions are satisfied for each $x \in X$:

(1) each $V_n(x)$ is a sequential neighborhood of x;

(2) $\{V_n(x) : n \in \mathbb{N}\}$ is a network at x;

(3) $V_{n+1}(x) \subseteq V_n(x)$ for each $n \in \mathbb{N}$.

Therefore, we will always assume that an *sn*-network of an *snf*-countable topological space satisfies the above conditions.

Lemma 3.1. *[28, Lemma 2.3] Suppose that $\{U_n : n \in \mathbb{N}\}$ is a decreasing countable network at x in X and W is a sequential neighborhood of x, then there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subseteq W$.*

The proof of the following lemma is direct.

Lemma 3.2. *Let G be an snf-countable semitopological group. Suppose that $\{V_n(x) : x \in G, n \in \mathbb{N}\}$ is an sn-network in G. For each $x \in G$ and $n \in \mathbb{N}$, put $W_n(x) = x \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is an sn-network in G.*

Lemma 3.3. *Let G be an snf-countable semitopological group with the sequentially continuous multiplication. Suppose that $\{V_n(x) : x \in G, n \in \mathbb{N}\}$ is an sn-network in G. For each $x \in G$ and $n \in \mathbb{N}$, put $W_n(x) = x \cdot V_n(e) \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is an sn-network in G.*

Proof. By Lemma 3.2, we can assume that $V_n(x) = x \cdot V_n(e)$ for each $x \in G$ and $n \in \mathbb{N}$. We will show that $\{W_n(e) : n \in \mathbb{N}\}$ is an *sn*-network at e in G. Actually, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_m(e) \subseteq V_n(e)$. Assuming the converse, take $x_m \in W_m(e) \setminus V_n(e)$ for each $m \in \mathbb{N}$, and let $x_m = a_m b_m$ where $a_m, b_m \in V_m(e)$. Since $\{V_n(e) : n \in \mathbb{N}\}$ is an *sn*-network at e in G, it follows that $a_m \to e$ and $b_m \to e$. Therefore, $x_m = a_m b_m \to e$ by the sequentially continuous multiplication of G. However, $V_n(e)$ is a sequential neighborhood of e in G, which is a contradiction. It shows that $\{W_n(e) : n \in \mathbb{N}\}$ is an *sn*-network at e in G, and $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is an *sn*-network in G by Lemma 3.2. □

The following theorem is the main result of this section.

Theorem 3.4. *Every snf-countable semitopological group G with the sequentially continuous multiplication is sof-countable.*

Proof. Let $\{V_n(x) : x \in G, n \in \mathbb{N}\}$ be an *sn*-network in G. For each $x \in G$ and $n \in \mathbb{N}$, we may assume that $V_n(x) = x \cdot V_n(e)$, by Lemma 3.2. Let $U_n = \{x \in V_n(e) : x \cdot V_k(e) \subseteq V_n(e)$ for some $k \in \mathbb{N}\}$. Obviously, $e \in U_n \subseteq V_n(e)$. Next we show that U_n is sequentially open in G. Indeed, take any $y \in U_n$, then $y \cdot V_k(e) \subseteq V_n(e)$ for some $k \in \mathbb{N}$. By Lemmas 3.1 and 3.3, it is easy to see that there exists $m \in \mathbb{N}$ such that $y \cdot (V_m(e) \cdot V_m(e)) \subseteq y \cdot V_k(e)$. Hence $(y \cdot V_m(e)) \cdot V_m(e) \subseteq V_n(e)$, which implies that $V_m(y) = y \cdot V_m(e) \subseteq U_n$. Since $V_m(y)$ is a sequential neighborhood of y, U_n is a sequential neighborhood of y. Therefore, the set U_n is sequentially open in G. It follows that $\{U_n : n \in \mathbb{N}\}$ is an *so*-network at e. Then G is sof-countable. □

Remark 3.5. F. Lin in [24] proved that every snf-countable paratopological group G is sof-countable, and posed the following question: let G be an snf-countable semitopological group or a quasitopological group. Is G sof-countable? Liu in [28] gave a negative answer to the question by constructing a Hausdorff weakly first-countable quasitopological group which is not sof-countable. It shows that the multiplication of G being sequentially continuous in Theorem 3.4 can not be omitted.

As immediate consequences we obtain the following corollaries, which improve the relevant results in [24].

Corollary 3.6. *Every sn-metrizable semitopological group G with the sequentially continuous multiplication is so-metrizable.*

Proof. Since G is *sn*-metrizable, it is easy to see that G is snf-countable. By Theorem 3.4, G is sof-countable. According to [26, Proposition 2.17], G is *so*-metrizable. The proof is completed. □

Corollary 3.7. *If G is a weakly first-countable semitopological group G with the sequentially continuous multiplication, then G is a first-countable paratopological group.*

Proof. Since a weakly first-countable space is snf-countable and sequential [25,37], it follows from Theorem 3.4 that G is sof-countable. Then G is first-countable because G is a sequential space.

We will show G is a paratopological group. Indeed, let $\{V_n : n \in \mathbb{N}\}$ be a decreasing open neighborhood base at e. By the proof of Lemma 3.3, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $V_m^2 \subseteq V_n$. Hence, the multiplication of G is jointly continuous. So G is a paratopological group. □

Corollary 3.8. *[32] If G is a weakly first-countable topological group, then G is metrizable.*

4. Some properties of quasitopological groups with sequentially continuous multiplications

First of all, we will construct a quasitopological group with the sequentially continuous multiplication which is not a topological group.

Suppose that X is a topological space, and G is an Abelian group. Let X^G be the space of all mappings of G to X, with the pointwise convergence topology (which in this case coincides with the Tychonoff product topology of X^G). For $a \in G, f \in X^G$ and each $x \in G$, put $s(a, f)(x) = s_a(f)(x) = f(x - a)$. Then $s_a(f) \in X^G$, and s is a mapping of $G \times X^G$ to X^G called the G-*shift* on X^G. The mapping $s_a : X^G \to X^G$ is called the a-*shift* of X^G, or the *shift* of X^G by a. For each $f \in X^G$, the subspace $s(G \times \{f\})$ of X^G is called the *orbit* of f under the shift s, or simply the *orbit* of f. A mapping $f : G \to X$ is called a *Korovin mapping* [6, p. 124] and the orbit of f is said to be a *Korovin orbit* [6, p. 124] if for every countable subset M of G and every mapping $h : M \to X$, there exists $a \in G$ such that $s_a(f)|_M = h$. For each $f \in X^G$ and $g \in G$, put $gf = s_g(f)$. We can define a group multiplication $*$ on the orbit of f, $Gf = \{gf : g \in G\}$ making

$$(g_1 f) * (g_2 f) = (g_1 g_2) f.$$

Then the mapping $k : G \to Gf$ given by $k(g) = gf$ is a homomorphism. In the case of a Korovin mapping, the homomorphism k is one-to-one. An important property of a Korovin mapping f is that the image of Gf under the natural projection of X^G onto X^B is the whole X^B, for any countable subset $B \subseteq G$ [6, Proposition 2.4.14].

We have following example by modifying Example 9 in [21].

Example 4.1. There exists a pseudocompact quasitopological group with the sequentially continuous multiplication which is not a topological group.

Proof. Let G be a Boolean group of the cardinality $\mathfrak{c}$ and X an infinite compact metrizable space. Then $|X| \leq \mathfrak{c}$ by [14, Theorem 3.1.29].

By [6, Theorem 2.4.13], there exists a Korovin mapping $f : G \to X$. Let $K = Gf$ be the Korovin orbit in X^G. Then K is a quasitopological group which fails to be a topological group by [6, Proposition 2.4.14]. According to [6, Theorem 2.4.15], K is pseudocompact.

We will show that the multiplication of K is sequentially continuous. It is enough to show that any countably infinite subspace of K is discrete.

Claim: Every countable infinite subspace of K is discrete.

Indeed, take an arbitrary countably infinite subset K_1 of K. Then there is a countably infinite subset H of G such that $K_1 = Hf$. Without loss of generality, we can assume that H is a subgroup of G. Since X is a non-discrete metrizable space, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ of pairwise distinct elements of X converging to $a \in X$. Let $Y = \{a\} \cup \{b_n : n \in \mathbb{N}\}$ and h be a one-to-one mapping of H to $Y \setminus \{a\}$. Since f is a Korovin mapping, there is $c \in G$ such that $s_c(f)|_H = h$. Therefore, $s_c(f)(e) = h(e)$, i.e., $f(e - c) = f(c) = h(e) = b_n$ for some $n \in \mathbb{N}$. Since b_n is an isolated point in Y, there is an open neighborhood W of b_n such that $Y \cap W = \{b_n\}$. Then $U = \{x \in X^G : x(c) \in W\}$ is open in X^G and $f \in U$. Take an arbitrary element $b \in H \setminus \{e\}$. Since h is one-to-one, we have $s_b(f)(c) = f(c - b) = f(b - c) = s_c(f)(b) = h(b) \neq h(e) = b_n$. Thus, $s_b(f) \notin U$. Clearly, each element of Hf has the form $s_b(f)$ for some $b \in H$. Therefore, $U \cap Hf = \{f\}$. Since f plays the role of the neutral element of K, the subgroup Hf of K is discrete.

By Claim, K has no non-trivial convergent sequence. Thus the multiplication of K is sequentially continuous. $\square$

Let $G = (\mathbb{R}^2, +)$ be the group with the usual pointwise addition. Clearly, G is an Abelian group. For each $p = (x_1, x_2) \in \mathbb{R}^2, \varepsilon > 0$, let

$$U(p, \varepsilon) = \{p\} \cup \{(x, y) : 0 < |x - x_1| < \varepsilon, |(y - y_1)/(x - x_1)| < \varepsilon\}.$$

We define a topology on G by giving a local base $\{U(p, \varepsilon) : \varepsilon > 0\}$ at each point $p \in G$. It is called the *bowtie topology* [19] on G. We denote by $\mathcal{D}$ the bowtie topology on G. $(G, \mathcal{D})$ is a completely regular quasitopological group [5].

Example 4.2. There exists a completely regular quasitopological group G such that the multiplication of G is not sequentially continuous.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the bowtie topology $\mathcal{D}$. Then $(G, \mathcal{D})$ is a completely regular first-countable quasitopological group. Clearly, the sequences $\{(\frac{1}{n}, \frac{1}{n^2})\}_{n\in\mathbb{N}}$ and $\{(-\frac{1}{n}, \frac{1}{n^2})\}_{n\in\mathbb{N}}$ converge to $e = (0, 0)$. However, the sequence $\{(\frac{1}{n}, \frac{1}{n^2}) + (-\frac{1}{n}, \frac{1}{n^2})\}_{n\in\mathbb{N}}$ is divergent. Thus the multiplication of G is not sequentially continuous. $\square$

If G is a quasitopological group, it is easy to check that σG is a quasitopological group. The following theorem improve the result in [24, Theorem 4.4].

Theorem 4.3. *Let G be an snf-countable quasitopological group with the sequentially continuous multiplication. Then σG is a topological group.*

Proof. It follows from Theorem 3.4 that G is sof-countable. Let $\{V_n : n \in \mathbb{N}\}$ be a decreasing so-network at e in G. Then $\{V_n : n \in \mathbb{N}\}$ is a neighborhood base at e in σG. In fact, let U be an open neighborhood of e in σG, then U is a sequentially open neighborhood of e in G. By Lemma 3.1 there exists $n \in \mathbb{N}$ such that $V_n \subseteq U$. Since G and σG have the same convergent sequences [8, p. 678], σG is a quasitopological group with the sequentially continuous multiplication. By Corollary 3.7, σG is a topological group. $\square$

Let S_κ be the quotient space obtained by identifying all limit points of the topological sum of κ many convergent sequences. S_ω is called the *sequential fan*. The *Arens' space* [14, Example 1.6.19] $S_2 = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ is defined as follows: each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m) : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\bigcup\{V_n : n > k\})$ for some $k \in \mathbb{N}$, where V_n is a neighborhood of x_n.

It was proved in [31] that a topological group contains a (closed) copy of S_ω if and only if it contains a (closed) copy of S_2. For quasitopological groups, we prove the following result which give a partial answer to Problem 1.1.

Theorem 4.4. *Let G be a sequential quasitopological group with the sequentially continuous multiplication. Then G contains a closed copy of S_ω if and only if it contains a closed copy of S_2.*

Proof. Sufficiency. Since G is a quasitopological group, without loss of generality, let $A = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ be a closed copy of S_2 in G, where e is the neutral element of G. For each $n, m \in \mathbb{N}$, let $y_n(m) = x_n^{-1} x_n(m)$, and put $S_n = \{y_n(m) : m \in \mathbb{N}\}$. Then $y_n(m) \to e$ as $m \to \infty$ for each $n \in \mathbb{N}$. For each m, $F = \{n : S_m \cap S_n$ is infinite$\}$ is finite (otherwise, pick distinct $x_{n_i}^{-1} x_{n_i}(m_i) \in S_m \cap S_{n_i}$ for $n_i \in F$ with $n_i < n_{i+1}$, then $x_{n_i}^{-1} x_{n_i}(m_i) \to e$ and $x_{n_i} \to e$. Hence $x_{n_i}(m_i) \to e$ by the sequentially continuous multiplication of G, a contradiction). Without loss of generality, we can assume $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $B = \{e\} \cup \{y_n(m) : n, m \in \mathbb{N}\}$.

Claim 1: B is a closed copy of S_ω in G.

Suppose B is not closed in G. Since G is sequential, there are $x \in \overline{B} \setminus B$ and an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\}$ of B such that $y_{n_i}(m_i) \to x$ as $i \to \infty$ and $n_i < n_{i+1}$. Since A is closed in G, there exists an open neighborhood V of e such that Vx meets $\{x_n(m) : m \in \mathbb{N}\}$ for at most one n. By the sequentially continuous multiplication of G and $x_n \to e$, we have $x_{n_i} y_{n_i}(m_i) \to x$. Since Vx is an open neighborhood of x, there are infinitely many $i \in \mathbb{N}$ with $x_{n_i}(m_i) \in Vx$, a contradiction.

If $f : \mathbb{N} \to \mathbb{N}$, then $C = \{y_n(m) : m \leq f(n), n \in \mathbb{N}\}$ does not have an accumulation point. Otherwise, if a is an accumulation point of C, there exist $x \in \overline{C \setminus \{a\}} \setminus (C \setminus \{a\})$ and an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\}$ of $C \setminus \{a\}$ such that $y_{n_i}(m_i) \to x$ as $i \to \infty$ and $n_i < n_{i+1}$. Let V be an open neighborhood of the neutral element e such that $|Vx \cap \{x_n(m) : m \leq f(n), n \in \mathbb{N}\}| \leq 1$. By the sequentially continuous multiplication of G and $x_n \to e$, we have $x_{n_i} y_{n_i}(m_i) \to x$. Since Vx is an open neighborhood of x, there are infinitely many $i \in \mathbb{N}$ with $x_{n_i}(m_i) \in Vx$, which is a contradiction. Hence B is a copy of S_ω.

Necessity. Let $A = \{e\} \cup \{y_n(m) : m, n \in \mathbb{N}\}$ be a closed copy of S_ω in G, for each n, $y_n(m) \to e$ as $m \to \infty$. Let U_n be an open neighborhood of $y_1(n)$ for each n with $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $x_n(m) = y_1(n) y_{n+1}(m)$ for each $n, m \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $x_n(m) \to y_1(n)$ as $m \to \infty$. Without loss of generality, we can assume $\{x_n(m) : m \in \mathbb{N}\} \subseteq U_n$. Let $B = \{e\} \cup \{y_1(n) : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$.

Claim 2: B is a closed copy of S_2 in G.

Suppose B is not closed. Then there exist $x \in \overline{B} \setminus B$ and an infinite subset $\{x_{n_i}(m_i) : i \in \mathbb{N}\}$ of B such that $x_{n_i}(m_i) \to x$ as $i \to \infty$ and $n_i < n_{i+1}$. Since A is closed, there is a neighborhood V of e such that $Vx \cap (A \setminus \{x\}) = \emptyset$. Note that G is a quasitopological group and $y_1(n) \to e$, we have $(y_1(n))^{-1} \to e$. By the sequentially continuous multiplication of G, $y_{n_i+1}(m_i) = (y_1(n_i))^{-1} x_{n_i}(m_i) \to x$ as $i \to \infty$. Since Vx is a neighborhood of x, Vx contains infinitely many elements of A, which is a contradiction.

If $f : \mathbb{N} \to \mathbb{N}$, similarly as in the proof of Claim 1, $\{x_n(m) : n \geq k, m \leq f(n)\}$ is closed for each $k \in \mathbb{N}$. Hence B is a copy of S_2. □

A countable collection $\{S_n : n \in \mathbb{N}\}$ of convergent sequences in a space X is called a *sheaf* (with a vertex x) if each sequence S_n converges to the same point $x \in X$. A space X is called an α_4-*space* [30], if for every point $x \in X$ and each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex x, there exists a sequence converging to x which meets infinitely many sequences S_n.

A space X is called a *strongly Fréchet* space [36] if for every decreasing sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets in X with $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, there is $x_n \in A_n$ for each $n \in \mathbb{N}$ such that the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x. It is known that a space is strongly Fréchet if and only if it is a Fréchet α_4-space [3,4]. Therefore, the following result is obtained.

Theorem 4.5. *Let G be a quasitopological group with the sequentially continuous multiplication. Then the following conditions are equivalent:*

(1) *G is a sequential α_4-space;*
(2) *G is Fréchet;*
(3) *G is strongly Fréchet.*

Proof. (1) ⇒ (2). Let $[A]$ be the set of all limit points of sequences in A for each $A \subseteq G$. Suppose G is not Fréchet. There is a subset A of G such that $[A] \neq \overline{A}$. If $[A]$ is closed in G, then $\overline{A} \subseteq \overline{[A]} = [A] \subseteq \overline{A}$, which is a contradiction. Hence, $[A]$ is not closed in G. Since G is sequential, $[A]$ is not sequentially closed. That is $[[A]] \neq [A]$. Thus there is $x \in [[A]] \setminus [A]$. By translating A through the multiplication by x^{-1}, we may assume $x = e$ without loss of generality.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of $[A]$ converging to e. For each x_n, let $\{x_n(j)\}_{j\in\mathbb{N}}$ be a sequence of points of A converging to x_n. Since the multiplication of G is separately continuous, $\{x_n^{-1}x_n(j)\}_{j\in\mathbb{N}}$ converges to e for each $n \in \mathbb{N}$. Since G is an α_4-space, it is possible to pick n_k, j_k for each $k \in \mathbb{N}$ such that $\{x_{n_k}^{-1}x_{n_k}(j_k)\}_{k\in\mathbb{N}}$ converges to e and $n_k < n_{k+1}$. By sequential continuity of the multiplication of G, $x_{n_k}(j_k) = x_{n_k}x_{n_k}^{-1}x_{n_k}(j_k) \to e$ as $k \to \infty$, contradicting the assumption that $e \notin [A]$. Therefore, G is Fréchet.

(2) $\Rightarrow$ (1). It is enough to show that for each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex e, there exists a sequence converging to e which meets infinitely many sequences S_n. For each $n \in \mathbb{N}$, let $S_n = \{x_n(j)\}_{j\in\mathbb{N}}$, the sequence $\{x_1(n)x_{n+1}(j)\}_{j\in\mathbb{N}}$ converges to $x_1(n)$. Put $A = \bigcup_{n\in\mathbb{N}}\{x_1(n)x_{n+1}(j) : j \in \mathbb{N}\}$, then $e \in \overline{A}$. By hypothesis there is a sequence S in A converging to e. Since G is Hausdorff, S intersects with infinitely many sequences $\{x_1(n)x_{n+1}(j)\}_{j\in\mathbb{N}}$. Let $S = \{x_1(n_i)x_{n_i+1}(j_i)\}_{i\in\mathbb{N}}$ where each $n_i < n_{i+1}$. Since G is a quasitopological group and $\{x_1(n)\}_{n\in\mathbb{N}}$ converges to e, $(x_1(n_i))^{-1} \to e$ as $i \to \infty$. By the sequentially continuous multiplication of G, $x_{n_i+1}(j_i) = (x_1(n_i))^{-1}x_1(n_i)x_{n_i+1}(j_i) \to e$ as $i \to \infty$.

Since a space is strongly Fréchet if and only if it is a Fréchet α_4-space [3,4], by (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (2) is obvious. □

It was showed in [32, Example 4] that neither (1) $\Rightarrow$ (2) nor (2) $\Rightarrow$ (1) in Theorem 4.5 can be extended to quasitopological groups.

Theorem 4.6. *Let G be a Fréchet quasitopological group with the sequentially continuous multiplication. If M is a first-countable space, then $G \times M$ is Fréchet.*

Proof. Take any subset A of $G \times M$ and any point $(x, y) \in \overline{A}$. Let p be the natural projection of $G \times M$ onto G. Fix a decreasing countable base $\{U_n : n \in \mathbb{N}\}$ of the point y in M, and put $B_n = p((G \times U_n) \cap A)$ for each $n \in \mathbb{N}$. Clearly, $x \in \overline{B_n}$. We also have $B_{n+1} \subseteq B_n$, since $U_{n+1} \subseteq U_n$. By Theorem 4.5, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ in G converging to x such that $b_n \in B_n$ for each $n \in \mathbb{N}$. Hence, there is $c_n \in U_n$ such that $(b_n, c_n) \in A$ for each $n \in \mathbb{N}$. Then the sequence $\{(b_n, c_n)\}_{n\in\mathbb{N}}$ converges to the point (x, y). □

Let (X, τ) be a space. A function $g : \mathbb{N} \times X \to \tau$ is called a *g-function* on X if, for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n + 1, x) \subseteq g(n, x)$. A space X is called a *q-space* [29] if there is a g-function on X satisfying for every sequence $\{x_n\}_{n\in\mathbb{N}}$ and a point p of X if $x_n \in g(n, p)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point. A space X is called a *β-space* [19, p. 475] if there is a g-function on X satisfying for every sequence $\{x_n\}_{n\in\mathbb{N}}$ and a point p of X if $p \in g(n, x_n)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point.

Since every first-countable quasitopological group G is semi-metrizable (see [22] or [35]), G is a β-space. Every first-countable space is a q-space. However, there are many countably compact topological groups which are not first-countable (see [6, Example 1.6.39 a)]). Since countably compact spaces are q-spaces, the following result generalizes [22, Theorem 2.5].

Theorem 4.7. *Let G be a quasitopological group. If G is a q-space, then G is a β-space.*

Proof. Since G is a q-space, there is a g-function $g : \mathbb{N}\times G \to \tau$ such that $x_n \in g(n, p)$ for each $n \in \mathbb{N}$ implies $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point. Define $g_1 : \mathbb{N}\times G \to \tau$ by $g_1(n, x) = xg(n, e)$ for each $n \in \mathbb{N}$ and $x \in G$, then g_1 is a g-function on G. If $p \in g_1(n, x_n)$ for each $n \in \mathbb{N}$, that is $p \in x_ng(n, e)$, thus $(x_n)^{-1}p \in g(n, e)$. By the definition of q-spaces, $\{(x_n)^{-1}p\}_{n\in\mathbb{N}}$ has an accumulation point. Since G is a quasitopological group, the inversion and translations of G are continuous. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point. Then G is a β-space. □

To conclude this section, we construct a non-metrizable, separable, normal and Moore quasitopological group under MA+¬CH.

Let τ, τ_1 be two topologies on X. We say that τ is regular with respect to τ_1 if for every $U \in \tau$ and $x \in U$ there is $V \in \tau$ such that $x \in V \subseteq \overline{V}^{\tau_1} \subseteq U$ [1].

Lemma 4.8. *[1] (MA) Let (X, τ) be a topological space which is the union of less than continuum compact subsets. If there is a weaker metric separable topology τ_1 such that τ is regular with respect to τ_1, then X^n is normal for every $n \in \mathbb{N}$.*

Example 4.9. (MA+¬CH) There exists a non-metrizable, separable, normal and Moore quasitopological group.

Proof. Let κ be a cardinal such that $\omega < \kappa < \mathfrak{c}$. Take a subset X of $\mathbb{R}$ such that $|X| = \kappa$ and $\mathbb{Q} \subseteq X$. Without loss of generality, we may assume that X is an additive subgroup of $\mathbb{R}$. Let $(G, \mathcal{D})$ be the bowtie space and $G_1 = \mathbb{Q} \times X$. Then $(G_1, \mathcal{D}|_{G_1})$ is a separable and Moore quasitopological group, see [35]. Clearly, $(G_1, \mathcal{D}|_{G_1})$ is not paracompact. It is easy to check that $\mathcal{D}|_{G_1}$ is regular with respect to $\mathcal{E}|_{G_1}$, where $\mathcal{E}$ is the Euclidean topology on $\mathbb{R}^2$. By Lemma 4.8, $(G_1, \mathcal{D}|_{G_1})$ is normal. □

It is known that under CH every separable normal Moore space is metrizable [20]. Thus under CH every separable, normal, Moore quasitopological group is metrizable. Therefore the existence of a separable, normal, non-metrizable Moore quasitopological group is independent of the usual axioms of Set theory.

5. Three-space problems in quasitopological groups

In this section, we first prove that hereditarily disconnectedness is a three-space property in semitopological groups. We also construct three examples to show that some three-space properties in topological groups cannot be extended to quasitopological groups.

For a topological space X we denote by $c_x(X)$ the connected component of X containing $x \in X$. A space X is called *hereditarily disconnected* [14, p. 360] if $c_x(X) = \{x\}$ for each $x \in X$. For a semitopological group G we denote by $c(G)$ the connected component of e and we call it briefly the connected component of G.

Proposition 5.1. Let G be a quasitopological group and N a closed normal subgroup of G. If both N and G/N are hereditarily disconnected, then so is G.

Proof. Let $q : G \to G/N$ be the canonical homomorphism. Assume that C is a connected subset in G. Then $q(C)$ is a connected subset of G/N, so by our hypothesis, $q(C)$ is a singleton. This means that C is contained in some coset xN. Since xN is hereditarily disconnected as well, we conclude that C is a singleton. Thus G is hereditarily disconnected. □

Example 5.2. There exists a completely regular first-countable quasitopological group G with a closed invariant subgroup H such that G/H and H are second-countable metrizable spaces, but G is not metrizable.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the bowtie topology. Then G is a completely regular quasitopological group. Define $f : G \to \mathbb{R}$ by $f(x, y) = y$ for each $(x, y) \in G$. Clearly, f is a continuous open homomorphism from G onto $\mathbb{R}$ with its usual topology and $H = \ker f = \mathbb{R} \times \{0\}$ carries the usual topology. According to [6, Theorem 1.5.3], G/H is topologically isomorphic to $\mathbb{R}$. So both H and G/H are second-countable metrizable spaces. However, G is not metrizable, since G is separable and contains an uncountable closed discrete subspace $\{0\} \times \mathbb{R}$. □

Remark 5.3. Obviously, Example 5.2 shows that metrizability is not a three-space property in quasitopological groups. It also shows that locally compact is not a three-space property. Clearly, both H and G/H are locally compact spaces. However, G is not locally compact, since every Hausdorff locally compact semitopological group is a topological group [13] and G is not a topological group. M. Choban showed that if H is a closed invariant subgroup of a topological group G such that H is second-countable and G/H has a countable network, then G has a countable network as well [38]. This result was extended to paratopological groups in [15, Theorem 2.6]. Example 5.2 also shows that the result cannot be extended to quasitopological groups, even if H is a second-countable topological group, since the quasitopological group G does not have a countable network.

Example 5.4. There exists a completely regular quasitopological group G with a closed invariant subgroup H such that G/H and H are second-countable, but G is not first-countable.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the topology $\mathcal{T}^*$ [5, p. 112]. Then G is a completely regular quasitopological group. Let $G_1 = \mathbb{Q}^2$ endowed with the subspace topology of G. Clearly, G_1 is completely regular, and G_1 is not first-countable by [5, Property 3.2]. Define $f : G_1 \to \mathbb{Q}$ by $f(x, y) = x$ for each $(x, y) \in G_1$. It is easy to check that f is a continuous open homomorphism from G_1 onto $\mathbb{Q}$ with its usual topology, and $H = \ker f = \{0\} \times \mathbb{Q}$ carries the discrete topology. Hence G_1/H is topologically isomorphic to $\mathbb{Q}$ by [6, Theorem 1.5.3]. So both H and G_1/H are second-countable spaces. However, G_1 is not first-countable. □

Remark 5.5. Example 5.4 shows that neither first-countability nor second-countability is a three-space property in quasitopological groups.

A.V. Arhangel'skiĭ and V.V. Uspenskij proved that if a topological group G contains a closed locally compact subgroup N and the quotient space G/N is paracompact, then so is G [7, Theorem 2.2]. In other words, paracompactness is an inverse invariant in topological groups under continuous open homomorphisms with locally compact kernels. Moreover, the quotient homomorphism $\pi : G \to G/N$ is locally perfect, i.e., there exists a closed neighborhood P of the neutral element in G such that $\pi|_P$ is a perfect mapping [6, Theorem 3.2.2]. These facts cannot be extended to paratopological groups. Indeed, it is shown in [23] that there exists a completely regular paratopological group G and an open continuous homomorphism f of G onto a paracompact paratopological group H with locally compact fibers such that G is not locally paracompact and f is not locally perfect. The following example shows that the results on topological groups ([7, Theorem 2.2] and [6, Theorem 3.2.2]) cannot be extended to quasitopological groups.

Example 5.6. There exists a completely regular quasitopological group G and an open continuous homomorphism f of G onto a paracompact quasitopological group H with locally compact fibers such that G is not locally paracompact and f is not locally perfect.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the bowtie topology $\mathcal{D}$. Then G is a completely regular quasitopological group. Put $U_n = \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 : n|y| < |x| < \frac{1}{n}\}$ for every $n \in \mathbb{N}$. It is easy to see that $\{U_n : n \in \mathbb{N}\}$ is a local base at the identity in $(G, \mathcal{D})$.

Let us show that G is not locally paracompact. Since paracompactness is hereditary with respect to closed subsets, we only need to show that the closure $\overline{U_n}$ of each basic open neighborhood U_n of the neutral $(0, 0)$ of G is not a paracompact subspace of G. Since $(\mathbb{Q} \times \mathbb{Q}) \cap \overline{U_n}$ is a countable dense subset of $\overline{U_n}$ and $(\{\frac{1}{2n}\} \times \mathbb{R}) \cap \overline{U_n}$ is a discrete closed subset of cardinality continuum in $\overline{U_n}$, it follows that $\overline{U_n}$ is not normal by [14, Corollary 2.1.10]. Hence, $\overline{U_n}$ is not a paracompact subspace of G. Thus G is not locally paracompact.

Define $f : G \to \mathbb{R}$ by $f(x, y) = x$ for each $(x, y) \in G$. Clearly, f is a continuous open homomorphism from G onto $\mathbb{R}$ with its usual topology, and $N = \ker f = \{0\} \times \mathbb{R}$ carries the discrete topology. Let $H = \mathbb{R}$ with the usual topology. Obviously, H is paracompact and N is locally compact. Since G is a quasitopological group, all the fibers of f are locally compact.

We will show that f is not locally perfect. In fact, for each $n \in \mathbb{N}$, the mapping $f|_{\overline{U_n}}$ is not a perfect mapping, since $\frac{1}{2n} \in f(\overline{U_n})$ and $f^{-1}(\frac{1}{2n}) \cap \overline{U_n}$ is not compact. This completes the proof. □

It is well known that pseudocompactness and precompactness are three-space properties in topological groups.

Question 5.7. *Is precompactness a three-space property in quasitopological groups?*

Addendum In the first draft of the paper, the authors did not know whether pseudocompactness is a three-space property in quasitopological groups. The reviewer pointed out that pseudocompactness is not a three-space property in quasitopological groups. Let $\mathscr{P}$ be a three-space property in quasitopological groups. If G and H satisfy $\mathscr{P}$, then $G \times H$ satisfies $\mathscr{P}$ as well. Since [21, Example 8] gave pseudocompact quasitopological groups G and H such that $G \times H$ is not pseudocompact, pseudocompactness is not a three-space property in quasitopological groups.

Acknowledgements

We wish to thank the reviewer for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper. In particular, the answer to Question 5.7 for pseudocompactness is due to the reviewer. The authors also would like to express their sincere appreciation to Professor M.G. Tkachenko for his help of constructing Example 4.1.

References

[1] K. Alster, T. Przymusiński, Normality and Martin's axiom, Fundam. Math. 91 (1974) 123–131.
[2] A.V. Arhangel'skiĭ, Mappings and spaces, Usp. Mat. Nauk 21 (4) (1966) 133–184 (in Russian).
[3] A.V. Arhangel'skiĭ, The frequency spectrum of topological spaces and the classification of spaces, Sov. Math. Dokl. 13 (1972) 265–268.
[4] A.V. Arhangel'skiĭ, The frequency spectrum of a topological space and the product operations, Trans. Mosc. Math. Soc. 2 (1981) 163–200.
[5] A.V. Arhangel'skiĭ, M.M. Choban, Examples of quasitopological groups, Bul. Acad. Ştiinţe Repub. Mold. Mat. 2(72)–3(73) (2013) 111–118.
[6] A.V. Arhangel'skiĭ, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Press, Amsterdam/Paris, 2008.
[7] A.V. Arhangel'skiĭ, V.V. Uspenskij, Topological groups: local versus global, Appl. Gen. Topol. 7 (1) (2006) 67–72.
[8] S. Baron, Sequential topologies, Am. Math. Mon. 73 (1966) 677–678.
[9] M. Bruguera, M. Tkachenko, The three space problem in topological groups, Topol. Appl. 153 (2006) 2278–2302.
[10] W.W. Comfort, L.C. Robertson, Extremal Phenomena in Certain Classes of Totally Bounded Groups, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa, 1988.
[11] W.W. Comfort, K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pac. J. Math. 16 (1966) 483–496.
[12] D.N. Dikranjan, V.V. Uspenskij, Categorically compact topological groups, J. Pure Appl. Algebra 126 (1998) 149–168.
[13] R. Ellis, Locally compact transformation groups, Duke Math. J. 24 (1957) 119–125.
[14] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, Berlin, 1989.
[15] M. Fernández, I. Sánchez, Three-space properties in paratopological groups, Topol. Appl. 216 (2017) 74–78.
[16] S.P. Franklin, Spaces in which sequences suffice, Fundam. Math. 57 (1965) 107–115.
[17] S.P. Franklin, Spaces in which sequences suffice II, Fundam. Math. 61 (1967) 51–56.
[18] Y. Ge, On sn-metrizable spaces, Acta Math. Sin. 45 (2002) 355–360 (in Chinese).
[19] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, 1984, pp. 423–501.
[20] R.W. Heath, Screenability, pointwise paracompactness and metrization of Moore spaces, Can. J. Math. 16 (1964) 763–770.
[21] C. Hernández, M. Tkachenko, Three examples of pseudocompact quasitopological groups, Topol. Appl. 153 (2006) 3615–3620.
[22] P. Li, L. Mou, On quasitopological groups, Topol. Appl. 161 (2014) 243–247.
[23] P. Li, L. Mou, S. Wang, Notes on questions about spaces with algebraic structures, Topol. Appl. 159 (17) (2012) 3619–3623.
[24] F. Lin, A note on paratopological groups with countable networks of sequential neighborhoods, Topol. Proc. 41 (2013) 9–16.

[25] S. Lin, On sequence-covering s-mappings, Adv. Math. (China) 25 (1996) 548–551 (in Chinese).
[26] S. Lin, Y. Ge, Preservations of so-metrizable spaces, Filomat 26 (4) (2012) 801–807.
[27] S. Lin, F.C. Lin, L.-H. Xie, The extensions of some convergence phenomena in topological groups, Topol. Appl. 180 (2015) 167–180.
[28] C. Liu, Paratopological (topological) groups with certain networks, Comment. Math. Univ. Carol. 55 (1) (2014) 111–119.
[29] E.A. Michael, A note on closed maps and compact sets, Isr. J. Math. 2 (1964) 173–176.
[30] T. Nogura, The product of $\langle\alpha_i\rangle$-spaces, Topol. Appl. 21 (3) (1985) 251–259.
[31] T. Nogura, D. Shakhmatov, Y. Tanaka, α_4-property versus A-property in topological spaces and groups, Studia Sci. Math. Hung. 33 (1997) 351–362.
[32] P.J. Nyikos, Metrizability and the Fréchet–Urysohn property in topological groups, Proc. Am. Math. Soc. 83 (1981) 793–801.
[33] L.-X. Peng, M.-Y. Guo, On reflections and three space properties of semi(para)topological groups, Topol. Appl. 217 (2017) 107–115.
[34] O. Ravsky, On H-closed paratopological groups, Visnyk Lviv. Univ. Ser. Mat.-Mekh. 59 (2001) 96–101.
[35] R.-X. Shen, On generalized metrizable properties in quasitopological groups, Topol. Appl. 173 (2014) 219.
[36] F. Siwiec, Sequence-covering and countably bi-quotient mappings, Gen. Topol. Appl. 1 (1971) 143–154.
[37] F. Siwiec, On defining a space by a weak base, Pac. J. Math. 52 (1974) 233–245.
[38] V.V. Uspenskij, Extensions of topological groups with a countable network, Mosc. Univ. Math. Bull. 39 (5) (1984) 84–85.
[39] L.-H. Xie, S. Lin, The extensions of paratopological groups, Topol. Appl. 180 (2015) 91–99.