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MP-equivalence of free paratopological groups ☆

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A B S T R A C T

Let $FP(X)$ denote the free paratopological group over a topological space X. Two topological spaces X and Y are called MP-*equivalent* if $FP(X)$ and $FP(Y)$ are topologically isomorphic. At first, it is shown that there exist non-homeomorphic topological spaces X and Y such that $FP(X)$ and $FP(Y)$ are topologically isomorphic. Secondly, MP-invariance of free paratopological groups is investigated. It is established that pseudocompactness, hereditary Lindelöfness, hereditary separability and the property of being a cosmic space are all MP-invariant, which generalizes some conclusions valid for free topological groups to free paratopological groups. Finally, a few questions about MP-equivalence of free paratopological groups are posed.

1. Introduction and preliminaries

A topological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous and the inverse mapping of G on itself is also continuous. In 1941, free topological groups in the sense of A. Markov were introduced [12]. M-equivalence of free topological groups were investigated in [2,7,13,15,20], etc. Two completely regular spaces X and Y are called M-equivalent (A-equivalent) [13] if $F(X)$ and $F(Y)$ ($A(X)$ and $A(Y)$) are topologically isomorphic, where $F(X)$ ($A(X)$) denotes the free (Abelian) topological group on X.

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A paratopological group is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. The absence of continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups. As a generalization of free topological groups, in 2002, S. Romaguera, M. Sanchis, and M. Tkachenko [19] introduced free paratopological groups on arbitrary topological spaces and discussed some of their topological properties.

Definition 1.1. [19] Let X be a subspace of a paratopological group G. Suppose that

(1) the set X generates G algebraically, that is, $\langle X \rangle = G$; and

(2) every continuous mapping $f : X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group* (briefly, *free paratopological group*) on X and is denoted by $FP(X)$.

If all groups in the above definition are Abelian, we obtain the definition of *Markov free Abelian paratopological group* (briefly, *free Abelian paratopological group*) on X, which is denoted by $AP(X)$.

Our main motivation to do this work arises from [2, Open Problem 7.4.4], posed by A. Arhangel'skiĭ and M. Tkachenko. This guides us to discuss which important results of free topological groups can be generalized to free paratopological groups. Around this subject, some publications about free paratopological groups have emerged, for example, see [3,4,9,10,16,17,19], etc.

In this paper, inspired by the concept M-equivalence of free topological groups, we shall reasonably introduce the notion of MP-equivalence of the free paratopological group $FP(X)$ on an arbitrary topological space X.

Definition 1.2. Two topological spaces X and Y are called *MP-equivalent* if $FP(X)$ and $FP(Y)$ are topologically isomorphic.

Definition 1.3. Two topological spaces X and Y are called *AP-equivalent* if $AP(X)$ and $AP(Y)$ are topologically isomorphic.

A topological property $\mathcal{P}$ is called *MP-invariant* (*AP-invariant*) if every topological space Y MP-equivalent (AP-equivalent) to a topological space X with $\mathcal{P}$ also has the property $\mathcal{P}$.

This paper is organized as follows.

At first, we shall show that there exist non-homeomorphic topological spaces X and Y such that $FP(X)$ and $FP(Y)$ are topologically isomorphic. Secondly, we make the first step towards the study of which topological properties are MP-invariant. We generalize a few results valid for free topological groups to free paratopological groups. Namely, we shall prove that pseudocompactness is an AP-invariant property and, a fortiori, MP-invariant property, and that $\mathcal{P}$ is MP-invariant if $\mathcal{P}$ is a hereditary, countably additive topological property. In particular, hereditary Lindelöfness, hereditary separability and the property of being a cosmic space are all MP-invariant. In addition, a few questions about MP-equivalence of free paratopological groups are posed.

In the paper, $F_a(X)$ ($A_a(X)$) denotes the algebraic free group (free Abelian group) on non-empty set X and e (0) is the identity of $F_a(X)$ ($A_a(X)$). The set X is called the free basis of $F_a(X)$ ($A_a(X)$). Here are some details, for instance, see [2]. Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $x_1, ..., x_n \in X$ and $\epsilon_1, ..., \epsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say in this case that the length $l(g)$ of g equals to n. Every element $g \in F_a(X)$ distinct from the identity e can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$ and $x_i \neq x_{i+1}$ for every $i = 1, ..., n-1$. Such an expression is called the *normal form* of g. Similar assertions (with the obvious changes for commutativity) are valid for $A_a(X)$.

Remark 1.4. It has been shown that the topology of $FP(X)$ ($AP(X)$) is the finest paratopological group topology on the group $F_a(X)$ ($A_a(X)$) which induces the original topology on X [19].

For every non-negative integer n, denote by $FP_n(X)$ ($AP_n(X)$) the subspace of the free paratopological group $FP(X)$ ($AP(X)$) that consists of all words of reduced length $\leq n$ with respect to the free basis X.

Remark 1.5. If X is a T_1-space, then $FP(X)$ is also T_1, X^{-1} is closed and discrete, and the subspaces X and $FP_n(X)$ of $FP(X)$ are all closed in $FP(X)$ for every non-negative integer n [3]. The same is true for $AP(X)$ [17].

In what follows, the subspace X of $FP(X)$ and $AP(X)$ is assumed to be T_1 in the paper. For some terminology unstated here, readers may refer to [2,5].

2. Existence of non-homeomorphic topological spaces X and Y such that $FP(X)$ and $FP(Y)$ are topologically isomorphic

Clearly, two homeomorphic topological spaces are MP-equivalent, but we shall prove in Theorem 2.7 that the converse fails to be true. Let us recall a few related notions. Let $\mathcal{P}$ be a cover of a topological space X. The space X is *determined* by $\mathcal{P}$ [8], or $\mathcal{P}$ is *generating* in X [2] if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is open (closed) in P for every $P \in \mathcal{P}$. If a topological space X can be expressed as be the union of an increasing sequence $\{X_n : n \in \mathbb{N}\}$ of its compact subsets X_n and is determined by $\{X_n : n \in \mathbb{N}\}$, then X is called a *k_ω-space* [6] and $X = \cup\{X_n : n \in \mathbb{N}\}$ is a *k_ω-decomposition* of X. It is well known that if both X and Y are Hausdorff k_ω-spaces, then $X \times Y$ is also a k_ω-space [11]. It is not difficult to verify that if $\cup\{X_n : n \in \mathbb{N}\}$ is a k_ω-decomposition of a topological space X and K is a compact subset of X, then $K \subset X_n$ for some n.

We need a few technical lemmas. Recall that A topological space X is called *functionally Hausdorff* if for any distinct points x and y of X, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. Obviously, every Tychonoff space is functionally Hausdorff and every functionally Hausdorff space is Hausdorff.

Lemma 2.1. *Let $\cup\{X_n : n \in \mathbb{N}\}$ be a k_ω-decomposition of a functionally Hausdorff countable k_ω-space X. Then $FP(X)$ is determined by $\{FP(X_n, X) : n \in \mathbb{N}\}$, where every $FP(X_n, X)$ denotes the subgroup of $FP(X)$ generated by X_n.*

Proof. We write $X = \{x_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let $H_n = \{x_i : i \leq n\}$ and $E_n = \{x_i^{-1} : i \leq n\}$. Obviously, for every $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $H_n \subset X_{m_n}$. Without loss of generality, we may assume $m_i < m_{i+1}$ for every $i \in \mathbb{N}$. Put $Y = X \cup X^{-1}$ and $Y_n = E_n \cup X_{m_n}$ for every $n \in \mathbb{N}$. The space X^{-1} being discrete by Remark 1.5, the space Y is determined by $\{Y_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, put

$$K_n = t_n((Y_n \cup \{e\})^n),$$

where t_n denotes the multiplication mapping of $(FP(X))^n$ to $FP(X)$. Then K_n is compact by the compactness of Y_n and the continuity of the mapping t_n. Obviously,

$$FP(X) = \cup\{K_n : n \in \mathbb{N}\}.$$

Claim. *The space $FP(X)$ is determined by $\{K_n : n \in \mathbb{N}\}$.*

Let τ be the original topology of the free paratopological group $FP(X)$. We define a new topology τ^* on the set $F_a(X)$ as follows. A subset O of $F_a(X)$ is in τ^* if and only if $O \cap K_n$ is open in K_n for every $n \in \mathbb{N}$, where every K_n carries the topology inherited from $(FP(X), \tau)$. Clearly, $\tau \subset \tau^*$ and $\tau|_{K_n} = \tau^*|_{K_n}$ for every $n \in \mathbb{N}$. We shall show that $(F_a(X), \tau^*)$ is a paratopological group, i.e., the multiplication mapping

$$op_2 : (F_a(X), \tau^*) \times (F_a(X), \tau^*) \to (F_a(X), \tau^*)$$

is continuous. Indeed, suppose C is an arbitrary compact subset of $(F_a(X), \tau^*) \times (F_a(X), \tau^*)$. Then $C \subset C_1 \times C_1$, where C_1 is some compact subset of $(F_a(X), \tau^*)$. Since $(F_a(X), \tau^*)$ is a k_ω-space, we have $C_1 \subset K_{n_0}$ for some n_0, whence

$$op_2(C) \subset op_2(C_1 \times C_1) \subset op_2(K_{n_0} \times K_{n_0}) \subset K_{2n_0}.$$

Since $(F_a(X), \tau)$ is a paratopological group, we have that the mapping

$$op_2|_{(C, \tau \times \tau|_C)} : (C, \tau \times \tau|_C) \to (K_{2n_0}, \tau|_{K_{2n_0}})$$

is continuous. By virtue of the equalities

$$\tau \times \tau|_C = \tau^* \times \tau^*|_C$$

and

$$\tau|_{K_{2n_0}} = \tau^*|_{K_{2n_0}},$$

the mapping

$$op_2|_{(C, \tau^* \times \tau^*|_C)} : (C, \tau^* \times \tau^*|_C) \to (K_{2n_0}, \tau^*|_{K_{2n_0}})$$

is continuous. Since X is functionally Hausdorff, $(FP(X), \tau)$ is Hausdorff by [17, Proposition 3.8]. Thus $(F_a(X), \tau^*) \times (F_a(X), \tau^*)$ is Hausdorff k_ω-space. By [5, Theorem 3.3.21], the mapping

$$op_2 : (F_a(X), \tau^*) \times (F_a(X), \tau^*) \to (F_a(X), \tau^*)$$

is continuous.

Now, suppose $A \subset Y$ and $A \in \tau^*|_Y$. Then there exists an open subset O of $(F_a(X), \tau^*)$ such that $A = O \cap Y$, whence $A \cap Y_n = O \cap Y \cap Y_n = O \cap Y_n \in \tau^*|_{Y_n} = \tau|_{Y_n}$ by $Y_n \subset K_n$ for every $n \in \mathbb{N}$. Hence $A \in \tau|_Y$ and $\tau^*|_Y = \tau|_Y$, and so $\tau^*|_X = \tau|_X$. By Remark 1.4, the topology τ of $FP(X)$ is the finest paratopological group topology on the group $F_a(X)$ which induces the original topology on X, so $\tau^* = \tau$. This completes the proof of the claim.

Clearly, $K_n \subset FP(X_{m_n}, X)$ for every $n \in \mathbb{N}$, so the space $FP(X)$ is determined $\{FP(X_{m_n}, X) : n \in \mathbb{N}\}$ by the above claim, equivalently, $FP(X)$ is determined by $\{FP(X_n, X) : n \in \mathbb{N}\}$. □

Remark 2.2. Lemma 2.1 improves [16, Theorem 2], one of the main results obtained by N. Pyrch in [16], which states that $FP(X)$ is a k_ω-space if X is a functionally Hausdorff countable k_ω-space.

The following two lemmas can be easily checked.

Lemma 2.3. *Suppose that both (G, τ_1) and (G, τ_2) are paratopological groups, $Y \subset G$ and $\tau_1|_Y = \tau_2|_Y$. Then for every $a \in G$, $\tau_1|_{aY} = \tau_2|_{aY}$.*

Lemma 2.4. *Let φ be a homomorphism of a group G onto a group H. If (G,τ) is a paratopological group, then (H,σ) is a paratopological group, where $\sigma = \{\varphi(V) : V \in \tau\}$.*

We introduce the notion of a topological basis of free paratopological groups, which is very useful for discussing MP-equivalence of free paratopological groups.

A subspace Y of the free paratopological group $FP(X)$ on a topological space X is called a *topological basis* of $FP(X)$ if Y is a free algebraic basis[1] of $FP(X)$ and the finest paratopological group topology on the abstract group $F_a(X)$ which induces on Y its original topology coincides with the topology of $FP(X)$. The Abelian case can be defined analogously. Especially, by Remark 1.4, the subspace X of $FP(X)$ ($AP(X)$) is a topological basis of $FP(X)$ ($AP(X)$).

Lemma 2.5. *If Y is a topological basis of the free paratopological group $FP(X)$ on a topological space X, then X and Y are MP-equivalent. The same is valid for the Abelian case.*

Proof. Let $i : Y \to FP(X)$ be the identity continuous mapping. We can extend the mapping i to a continuous homomorphism $\hat{i} : FP(Y) \to FP(X)$. Since Y is a free algebraic basis of $FP(X)$, $\hat{i} : FP(Y) \to FP(X)$ is an isomorphism. Let τ and $\mathcal{T}$ denote the original topologies of $FP(Y)$ and $FP(X)$ respectively. We define a new topology $\sigma = \{\hat{i}(U) : U \in \tau\}$ on the abstract group $F_a(X)$. It follows from Lemma 2.4 that $(F_a(X), \sigma)$ is a paratopological group. It is easy to see that $\hat{i} : (FP(Y), \tau) \to (F_a(X), \sigma)$ is a homeomorphism. Further, $\tau|_Y = \sigma|_Y$, i.e., σ induces on Y its original topology. The space Y being a topological basis of $FP(X)$, we have $\mathcal{T} \supset \sigma$, whence $\hat{i} : (FP(Y), \tau) \to (FP(X), \mathcal{T})$ is an open mapping. Hence, $\hat{i} : (FP(Y), \tau) \to (FP(X), \mathcal{T})$ is topologically isomorphic, i.e., X and Y are MP-equivalent. The arguments are valid for the Abelian case. □

Remark 2.6. Every topological basis Y of the free paratopological group $FP(X)$ on a topological space X is closed in $FP(X)$. The same is valid for the Abelian case. Indeed, it follows from Lemma 2.5 that the continuous identity mapping $i : Y \to FP(X)$ can extend to a topological isomorphism $\hat{i} : FP(Y) \to FP(X)$. By Remark 1.5, Y is closed in $FP(Y)$, whence $\hat{i}(Y) = Y$ is closed in $FP(X)$. The arguments are valid for the Abelian case.

Let $X = \{0\} \cup \mathbb{N}^2$. $\mathbb{N}^{\mathbb{N}}$ denotes the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. For every $n, m, k \in \mathbb{N}$, put $W(n,m) = \{(n,k) : k \geq m\}$. For every $x \in \mathbb{N}^2$, let $\mathcal{B}(x) = \{\{x\}\}$. Let

$$\mathcal{B}(0) = \{\{0\} \cup \bigcup_{n\in\mathbb{N}} W(n, f(n)) : f \in \mathbb{N}^{\mathbb{N}}\}.$$

The topological space X, generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x\in X}$, is called *countable fan* space and denoted by S_ω [1].

Theorem 2.7. *Let $\{C_i : i \in \mathbb{N}\}$ be a countable family of pairwise disjoint convergence sequences $C_i = \{x_i\} \cup \{x_{i,j} : j \in \mathbb{N}\}$ homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of the real line $\mathbb{R}$, where $\{x_{i,j}\}_{j\in\mathbb{N}}$ converges to x_i. Then the topological sum $X = \bigoplus_{i\in\mathbb{N}} C_i$ and the countable fan space S_ω are MP-equivalent.*

Proof. The space X is a metrizable countable k_ω-space with the k_ω-decomposition $X = \cup\{D_n : n \in \mathbb{N}\}$, where $D_n = \bigcup_{i\leq n} C_i$ for every $n \in \mathbb{N}$. By Lemma 2.1, $FP(X)$ is determined by $\{FP(D_n, X) : n \in \mathbb{N}\}$. Put

[1] Y is a *free algebraic basis* of $FP(X)$ if Y generates $F_a(X)$ algebraically and there are no algebraic relations in $F_a(X)$ between elements of the set Y.

$$Y = \{x_i : i \in \mathbb{N}\} \cup \bigcup_{i \in \mathbb{N}} x_1 x_i^{-1} C_i.$$

Clearly, Y is also a free algebraic basis of $FP(X)$ since X is a free algebraic basis of $FP(X)$. We shall show that Y is a topological basis of $FP(X)$. Let τ' be the finest paratopological group topology on the abstract group $F_a(X)$ which induces on Y its original topology. Clearly, τ' is finer than the original topology τ of $FP(X)$ and $\tau'|_Y = \tau|_Y$. By Lemma 2.3, $\tau'|_{C_i} = \tau|_{C_i}$ for every $i \in \mathbb{N}$. Since $X = \bigoplus_{i\in\mathbb{N}} C_i$, $id_X : (X, \tau|_X) \to (X, \tau'|_X)$ is continuous, which implies $\tau'|_X = \tau|_X$. By Remark 1.4, $\tau' = \tau$, i.e., Y is a topological basis of $FP(X)$. It follows from Lemma 2.5 that X and Y are MP-equivalent. It remains to verify that Y is homeomorphic to S_ω.

For every $i \in \mathbb{N}$, let $L_i = x_1 x_i^{-1} C_i$. Put

$$Y_0 = \bigcup_{i \in \mathbb{N}} L_i.$$

Since $FP(X)$ is a paratopological group, L_i converges to x_1 for every $i \in \mathbb{N}$. Clearly

$$L_i \cap L_j = \{x_1\}$$

for any distinct $i, j \in \mathbb{N}$. Since X is functionally Hausdorff, $FP(X)$ is Hausdorff [17, Proposition 3.8]. Hence

$$Y_0 \cap FP(D_n, X) = \bigcup_{i \le n} L_i$$

is compact and so closed in $FP(X)$ for every $n \in \mathbb{N}$. Then Y_0 is closed in $FP(X)$ because $FP(X)$ is determined by $\{FP(D_n, X) : n \in \mathbb{N}\}$. Also, it follows that the space Y_0 is determined by $\{\bigcup_{i\le n} L_i : n \in \mathbb{N}\}$. So the space Y_0 is homeomorphic to S_ω.

By Remark 1.5, $(X \setminus \{x_{1,j} : j \in \mathbb{N}\}) \cap Y = \{x_i : i \in \mathbb{N}\}$ is closed in Y. Obviously,

$$Y_0 \cap \{x_i : i \in \mathbb{N}\} = \{x_1\}.$$

Since the subspace $D = \{x_i : i \in \mathbb{N}\}$ of the space X is a discrete space, we have

$$Y = Y_0 \oplus \{x_i : i \ge 2\}.$$

Therefore, the space Y is homeomorphic to S_ω. □

Remark 2.8. It is easy to verify that the space S_ω is neither locally compact nor first-countable. Therefore, MP-equivalence does not preserve local compactness, first-countability or metrizability.

3. MP-invariance of free paratopological groups

In the section, we shall mainly investigate MP-invariance of pseudocompactness, hereditary Lindelöfness, hereditary separability and the property of being a cosmic space.

First of all, we shall show that MP-equivalent topological spaces are always AP-equivalent.

Lemma 3.1. *[2, Proposition 1.5.12] Let G and H be paratopological groups, let $p : G \to H$ be a topological isomorphism. If G_0 is an invariant subgroup of G and $H_0 = p(G_0)$, then the quotient groups G/G_0 and H/H_0 are topologically isomorphic.*

Lemma 3.2. *[2, Theorem 1.5.13] Let G and H be paratopological groups, $p : G \to H$ be an open continuous surjective homomorphism and N be the kernel of the homomorphism p. Then the mapping $\phi : G/N \to H$ which assigns to a coset xN the element $p(x)$ is a topological isomorphism.*

Theorem 3.3. *MP-equivalent topological spaces X and Y are AP-equivalent.*

Proof. Suppose there exists a topological isomorphism $h : FP(X) \to FP(Y)$. Denote by K_X and K_Y the derived subgroup[2] of $FP(X)$ and $FP(Y)$, respectively. Obviously $h(K_X) = K_Y$. Thus, by Lemma 3.1, the quotient groups $FP(X)/K_X$ and $FP(Y)/K_Y$ are topologically isomorphic. Now, in order to prove this theorem, it suffices to show the following claim.

Claim. *$AP(X)$ is topologically isomorphic to the quotient group $FP(X)/K_X$.*

Indeed, let $id_X : X \to X$ be identity mapping, and then we extend the mapping id_X to the continuous surjective homomorphism $\varphi : FP(X) \to AP(X)$. Clearly, the kernel of the homomorphism φ coincides with K_X. We shall show that φ is an open mapping. According to Lemma 2.4, we define a new paratopological group topology

$$\sigma = \{\varphi(U) : U \text{ is open in } FP(X)\}$$

on the set $A_a(X)$. Since $\varphi : FP(X) \to AP(X)$ is continuous, σ is finer than the original topology τ of $AP(X)$. Let U be an open subset of $FP(X)$ and assume that $x \in \varphi(U) \cap X$. Pick $z \in U$ such that $\varphi(z) = x$. Then $W = X \cap xz^{-1}U$ is an open neighbourhood of x in X and $W = \varphi(W) \subset \varphi(U) \cap X$. Hence, $\varphi(U) \cap X$ is open in X, and so σ induces on X its original topology. Further, by Remark 1.4, $\sigma = \tau$, and then φ is an open mapping. Thus the quotient group $FP(X)/K_X$ is topologically isomorphic to $AP(X)$ by Lemma 3.2. □

However, we do not know the answer to the following question.

Question 3.4. *Is it true that AP-equivalence does not imply MP-equivalence?*

Next, we shall show that pseudocompactness is an AP-invariant property and, a fortiori, MP-invariant property.

The following lemma can be directly checked according to the definition of a topological basis.

Lemma 3.5. *Let X and Y be topological spaces. If $\varphi : FP(X) \to FP(Y)$ is a topological isomorphism, then $\varphi^{-1}(Y)$ is a topological basis for $FP(X)$. The same is valid for the Abelian case.*

Recall that a completely regular space X is called *pseudocompact* [5] if every continuous real-valued function defined on X is bounded.

Theorem 3.6. *Suppose two completely regular spaces X and Y are AP-equivalent, more generally, MP-equivalent. If Y is pseudocompact, then so is X.*

Proof. Let $\varphi : AP(X) \to AP(Y)$ be a topological isomorphism. Then $\varphi^{-1}(Y)$ is a topological basis for $AP(X)$ by Lemma 3.5. For the sake of brevity, without loss of generality, we assume that Y is a topological basis for $AP(X)$. Suppose that X is not pseudocompact. Then it is easy to check that X contains a discrete family $\{U_i : i \in \omega\}$ of non-empty open subsets of X. For every $i \in \omega$, pick a point $x_i \in U_i$.

[2] The *derived subgroup* G' [18] of an abstract group G is the subgroup of G generated by all commutators $x^{-1}y^{-1}xy$, where $x, y \in G$. The derived subgroup G' of G is an invariant subgroup of G.

For $g \in AP(X)$ and $x \in X$, denote by $c(x, g)$ the coefficient k that stands at x in the normal form of g with respect to the free basis X. In other words, if $g = kx + k_1x_1 + \cdots + k_nx_n$, where $x, x_1, ..., x_n$ are pairwise distinct elements of X and $k, k_1, ..., k_n \in \mathbb{Z}$, then $c(x, g) = k$. In particular, $c(x, g) = 0$ if and only if x does not appear in the normal form of g.

Since Y is a free algebraic basis for $AP(X)$, there exists $y_i \in Y$ such that $c(x_i, y_i) \neq 0$ for every $i \in \omega$. Let $t_{i,1}, ..., t_{i,n_i} \in X$ be all letters distinct from x_i which appear in the normal form of y_i (possibly, $n_i = 0$). Without loss of generality, we can assume that $c(x_j, y_i) = 0$ whenever $i < j$.

By induction on $n \in \omega$, we define continuous real-valued functions f_n on X as follows. Put $f_0 \equiv 0$. Suppose that for some $n \geq 1$, we have defined the functions $f_0, ..., f_{n-1}$. Put $g_n = \sum_{i=0}^{n-1} f_i$. Let

$$F_n = (X \setminus U_n) \cup \{t_{i,j} : i \leq n, j \leq n_i, t_{i,j} \in U_n\}.$$

Clearly, F_n is closed in X, and $x_n \notin F_n$ by $c(x_j, y_i) = 0$ whenever $i < j$. Since X is completely regular, there exists a continuous real-valued function f_n on X such that $f_n(F_n) \subset \{0\}$ and

$$f_n(x_n) = n + \sum_{j \in \{j:\ t_{n,j} \in U_0 \cup \cdots \cup U_{n-1}\}} |c(t_{n,j}, y_n) g_n(t_{n,j})|.$$

This completes our construction. Since $\{U_i : i \in \omega\}$ is discrete in X, the function $f = \sum_{n \in \omega} f_n$ is continuous by [5, Corollary 2.1.12]. It is easy to check that the following hold.

(1) For every $n \in \omega$, $f(x_n) = f_n(x_n)$.

(2) For every $i \geq 1$ and $j \leq n_i$, $f(t_{i,j}) = 0$ whenever $t_{i,j} \notin U_0 \cup \cdots \cup U_{i-1}$.

(3) For every $i \geq 1$ and $j \leq n_i$, $f(t_{i,j}) = g_i(t_{i,j})$ whenever $t_{i,j} \in U_0 \cup \cdots \cup U_{i-1}$.

Now, we extend f to a continuous homomorphism $\psi : AP(X) \to \mathbb{R}$. For every $i \geq 1$, since

$$y_i = c(x_i, y_i)x_i + c(t_{i,1}, y_i)t_{i,1} + \cdots + c(t_{i,n_i}, y_i)t_{i,n_i},$$

we have

$$\begin{aligned} |\psi(y_i)| &= |c(x_i, y_i)f(x_i) + c(t_{i,1}, y_i)f(t_{i,1}) + \cdots + c(t_{i,n_i}, y_i)f(t_{i,n_i})| \\ &= |c(x_i, y_i)(i + \sum_{j \in \{j:\ t_{i,j} \in U_0 \cup \cdots \cup U_{i-1}\}} |r_{i,j}|) + \sum_{j \in \{j:\ t_{i,j} \in U_0 \cup \cdots \cup U_{i-1}\}} r_{i,j}| \\ &\geq i, \end{aligned}$$

where

$$r_{i,j} = c(t_{i,j}, y_i) g_i(t_{i,j}).$$

Because $\{y_i : i \geq 1\} \subset Y$, the space Y is not pseudocompact. This is a contradiction. □

The following questions arise naturally from our investigations.

Question 3.7. *Suppose two completely regular spaces X and Y are AP-equivalent or MP-equivalent. Is Y compact if X is compact?*

Question 3.8. *Suppose two completely regular spaces X and Y are AP-equivalent or MP-equivalent. Is Y countably compact if X is countably compact?*

However, we shall prove that another covering property, hereditary Lindelöfness, is MP-invariant. Indeed, we can obtain a more general theorem.

Lemma 3.9. *[10, Proposition 6.3] Let X be a topological space and i_n denote the continuous multiplication mapping of $\tilde{X}^n$ onto $FP_n(X)$ for every $n \in \mathbb{N}$, where $\tilde{X} = X \oplus \{e\} \oplus X^{-1}$. Put $C_n(X) = FP_n(X) \setminus FP_{n-1}(X)$ and $C_n^*(X) = i_n^{-1}(C_n(X))$ for $n \geq 1$. Then the mapping i_n homeomorphically maps $C_n^*(X)$ onto $C_n(X)$ for every $n \in \mathbb{N}$.*

The following fact about general topology is evident.

Lemma 3.10. *Let both $f : X_1 \to Y_1$ and $g : Y_2 \to X_2$ be continuous mappings between topological spaces, where $X_1 \subset X_2$ and $Y_1 \subset Y_2$. If $F = \{x \in X_1 : g(f(x)) = x\} \neq \emptyset$, then $f|_F : F \to f(F)$ is a homeomorphism.*

Lemma 3.11. *If X and Y are MP-equivalent topological spaces, then Y can be represented as the union of countably many subspaces each of which is homeomorphic to a subspace of X.*

Proof. Let $\varphi : FP(X) \to FP(Y)$ be a topological isomorphism. Then $\varphi^{-1}(Y)$ is a topological basis for $FP(X)$ by Lemma 3.5. For the sake of brevity, without loss of generality, we assume that Y is a topological basis for $FP(X)$. For $g \in FP(X)$, denote by $l_Y(g)$ the reduced length of g with the respect to the free algebraic basis Y. For $n \in \mathbb{N}$ and $\nu = (m_1, ..., m_n) \in \mathbb{N}^n$, put

$$C_n(Y) = \{g \in FP(X) : l_Y(g) = n\},$$

and

$$Y_\nu = Y \cap C_n(X) \cap ((\tilde{X} \cap C_{m_1}(Y)) \cdots (\tilde{X} \cap C_{m_n}(Y))),$$

where $\tilde{X} = X \oplus \{e\} \oplus X^{-1}$ and $C_n(X) = FP_n(X) \setminus FP_{n-1}(X)$. Clearly, $Y = \bigcup_{\nu \in \Xi} Y_\nu$, where $\Xi = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$.

Let $n \in \mathbb{N}$ and $\nu = (m_1, ..., m_n) \in \mathbb{N}^n$ be arbitrary. For every $i = 1, ..., n$, denote by φ_i the mapping from Y_ν to X assigning to $g \in Y_\nu$ the point of X that appears at the i-th place in the reduced form of g with respect to the free algebraic basis X. Analogously, for every $j = 1, ..., m_i$, denote by $\psi_{i,j}$ the mapping from $C_{m_i}(Y)$ to Y assigning to an element $h \in C_{m_i}(Y)$ the point of Y that appears at the j-th place in the reduced form of h with respect to the free algebraic basis Y. By Lemma 3.9, the mappings φ_i and $\psi_{i,j}$ are continuous. Let

$$F_{i,j}^\nu = \{w \in Y_\nu : \psi_{i,j}(\varphi_i(w)) = w\}.$$

Then Lemma 3.10 implies that every $F_{i,j}^\nu$ is homeomorphic to a subspace of X. It remains to verify that

$$Y = \cup\{F_{i,j}^\nu : \nu = (m_1, ..., m_n) \in \mathbb{N}^n, n \in \mathbb{N}, i = 1, ..., n, j = 1, ..., m_i\}.$$

Indeed, suppose that $y \in Y_\nu$, where $\nu = (m_1, ..., m_n) \in \mathbb{N}^n$. Then we can write

$$y = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n},$$

and for every $i = 1, ..., n$,

$$x_i = y_{i,1}^{\delta_{i,1}} \cdots y_{i,m_i}^{\delta_{i,m_i}},$$

where $x_i \in X$, $\varepsilon_i = \pm 1$, $y_{i,j} \in Y$ and $\delta_{i,j} = \pm 1$. Thus

$$y = (y_{1,1}^{\delta_{1,1}} \cdots y_{1,m_1}^{\delta_{1,m_1}})^{\varepsilon_1} \cdots (y_{n,1}^{\delta_{n,1}} \cdots y_{n,m_n}^{\delta_{n,m_n}})^{\varepsilon_n}.$$

Since Y is a free algebraic basis for $FP(X)$, there exist some i and j such that $y = y_{i,j}$. This implies that

$$\psi_{i,j}(\varphi_i(y)) = \psi_{i,j}(x_i) = y_{i,j} = y,$$

and so $y \in F^{\nu}_{i,j}$. This completes the proof. □

A topological property $\mathcal{P}$ is said to *countably additive* if every space X, which can be expressed as the union of a countable family of its subspaces X_n with the property $\mathcal{P}$, has also $\mathcal{P}$. Lemma 3.11 immediately implies the following theorem.

Theorem 3.12. *Assume that $\mathcal{P}$ is a hereditary, countably additive topological property. Let X and Y be MP-equivalent topological spaces. If X has the property $\mathcal{P}$, then so does Y. Namely, $\mathcal{P}$ is MP-invariant.*

A topological space X is called a *cosmic* space [14] if X has a countable network. Theorem 3.12 leads to the following corollary.

Corollary 3.13. *Let X and Y be MP-equivalent topological spaces. Then the following hold.*

(1) If X is hereditarily Lindelöf, then so is Y.
(2) If X is hereditarily separable, then so is Y.
(3) If X is a cosmic space, then so is Y.

Question 3.14. *Is valid Corollary 3.13 for the AP-equivalent case?*

Question 3.15. *Is Lindelöfness MP-invariant or AP-invariant?*

Question 3.16. *Is separability MP-invariant or AP-invariant?*

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References

[1] A. Arhangel'skiĭ, S. Franklin, Ordinal invariants for topological spaces, Mich. Math. J. 15 (1968) 313–320.
[2] A. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Scientific, 2008.
[3] A. Elfard, P. Nickolas, On the topology of free paratopological groups, Bull. Lond. Math. Soc. 44 (6) (2012) 1103–1115.
[4] A. Elfard, P. Nickolas, On the topology of free paratopological groups II, Topol. Appl. 160 (2013) 220–229.
[5] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989 (revised and completed edition).
[6] S. Franklin, B. Thomas, A survey of k_ω-spaces, Topol. Proc. 2 (1977) 111–124.
[7] M. Graev, Free topological groups, Dokl. Akad. Nauk SSSR 12 (1948) 279–324 (in Russian).
[8] G. Gruenhage, E. Michael, Y. Tanaka, Spaces determined by point-countable covers, Pac. J. Math. 113 (2) (1984) 303–332.
[9] F. Lin, A note on free paratopological groups, Topol. Appl. 159 (2012) 3596–3604.
[10] F. Lin, C. Liu, S. Lin, S. Cobzas, Free Abelian paratopological groups over metric spaces, Topol. Appl. 183 (2015) 90–109.
[11] S. Lin, Point-Countable Covers and Sequence-Covering Mappings, Science Press, 2002 (in Chinese).
[12] A. Markov, On free topological groups, Dokl. Akad. Nauk SSSR 31 (1941) 299–301 (in Russian).
[13] A. Markov, On free topological groups, Izv. Akad. Nauk SSSR 9 (1945) 3–64 (in Russian).
[14] E. Michael, $\aleph_0$-spaces, J. Math. Mech. 15 (1966) 983–1002.
[15] V. Pestov, Some topological properties preserved by the relation of M-equivalence, Russ. Math. Surv. 39 (1984) 223–224.
[16] N. Pyrch, Free paratopological groups and free products of paratopological groups, J. Math. Sci. 174 (2011) 190–195.
[17] N. Pyrch, O. Ravsky, On free paratopological groups, Mat. Stud. 25 (2006) 115–125.
[18] D. Robinson, A Course in the Theory of Groups, Springer-Verlag Inc., New York, 1982.
[19] S. Romaguera, M. Sanchis, M. Tkachenko, Free paratopological groups, Topol. Proc. 27 (2002) 1–28.
[20] M. Tkachenko, Generalization of a theorem of Comfort and Ross I, Ukr. Mat. Zh. 41 (3) (1989) 377–382 (in Russian).