Topology and its Applications 209 (2016) 198–206

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On the countable tightness and the k-property of free topological groups over generalized metrizable spaces ☆

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ARTICLE INFO

Article history:
Received 3 February 2016
Received in revised form 4 June 2016
Accepted 14 June 2016
Available online 17 June 2016

MSC:
54H11
54D50
54D55
54E20
54E99
22A05

Keywords:
Free topological groups
k-Spaces
Countable tightness
k_ω-Spaces
σ-Spaces
k^*-Metrizable spaces
k-Semistratifiable spaces

ABSTRACT

In this paper, the countable tightness and the k-space property in free topological groups over generalized metrizable spaces are considered. The following main results are obtained:
(1) A space X is separable or discrete if and only if the free topological group $F(X)$ is of countable tightness for a paracompact σ-, k-space X;
(2) A space X is an $\aleph_0$-space or discrete if and only if $F(X)$ is of countable tightness for a normal k^*-metrizable k-space X;
(3) A space X is a k_ω-space or discrete if and only if $F(X)$ is a k-space for a k^*-metrizable space X.
Some results on the countable tightness and the k-property of free topological groups over metric spaces are improved.

☆ This research is supported by NSFC (Nos. 11526158, 11471153, 11271262, 11501404), the PhD Start-up Fund of Natural Science Foundation of Guangdong Province (Nos. 2014A030310187, 2016A030310002), the Project of Young Creative Talents of Guangdong Province (Nos. 2014KQNCX161, 2015KQNCX171), the Natural Science Foundation of Hebei Province (A2014501101) and the Foundation of Hebei Educational Committee (QN2015303), the Natural Science Foundation of Wuyi University of China (No. 2015zk07), and Jiangmen science and technology plan projects (No. 201501003001412).

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http://dx.doi.org/10.1016/j.topol.2016.06.009
0166-8641/

1. Introduction

The symbols $F(X)$ and $A(X)$ denote respectively the *free topological group* and the *free Abelian topological group* on a Tychonoff space X in the sense of Markov [21]. The topology of $F(X)$ is rather complicated even for a very simple space X. Indeed, only when X is discrete can the space $F(X)$ have the Baire property or the Fréchet–Urysohn property [2]. Neither $F(\mathbb{Q})$ nor $A(\mathbb{Q})$ are k-spaces [7], where $\mathbb{Q}$ stands for all rational numbers. The following is a famous theorem obtained by J. Mack, S.A. Morris and E.T. Ordman.

Theorem 1.1 *([20]). A space X is a k_ω-space if and only if the space $F(X)$ (or $A(X)$) is a k_ω-space.*

Thus the following question was considered by several topologists [2,3,11,12,27,28].

Question 1.2 *([2]). When, in terms of the space X, is the space $F(X)$ (or $A(X)$) a k-space or of countable tightness?*

Arhangel'skiĭ, Okunev and Pestov [2] gave the characterizations of the countable tightness or k-space property of $F(X)$ on a metrizable space X as follows.

Theorem 1.3 *([2, Theorem 3.6]). The tightness of $F(X)$ is countable if and only if X is either separable or discrete for a metrizable space X.*

Theorem 1.4 *([2, Theorem 3.7]). The following are equivalent for a metrizable space X:*

(1) *$F(X)$ is a k-space;*
(2) *$F(X)$ is a k_ω-space;*
(3) *X is locally compact separable or discrete.*

Recently, Z. Li, F. Lin and C. Liu [11] attempted to extend Theorems 1.3 and 1.4 to some generalized metrizable spaces. For example,

Theorem 1.5 *([11, Proposition 4.4]). The tightness of $F(X)$ is countable if and only if X is separable or discrete for a stratifiable k-space X.*

Theorem 1.6 *([11, Theorem 4.11]). $F(X)$ is a k-space if and only if X is a k_ω-space or discrete for a k^*-metrizable μ-space X.*

In this paper, we characterize some generalized metrizable spaces X such that $F(X)$ is of countable tightness or a k-space. The paper is organized as follows. Section 2 introduces terminology and notation used throughout the paper. In Section 3, some basic facts are established, which play an important role in the main body of the article. The countability of tightness in $F(X)$ are studied in Section 4. We mainly shows that: $F(X)$ is of countable tightness if and only if X is separable or discrete for a paracompact σ-, k-space (see Theorem 4.1). Further, $F(X)$ is of countable tightness if and only if X is an $\aleph_0$-space or discrete for a normal k^*-metrizable k-space X (see Theorem 4.3). In Section 5, we characterize some generalized metrizable spaces X such that $F(X)$ is a k-space. It is shown that: $F(X)$ is a k-space if and only if X is locally compact separable metrizable or discrete for a first-countable paracompact σ-space X (see Theorem 5.2), and $F(X)$ is a k-space if and only if X is locally k_ω separable or discrete for a k^*-metrizable space X (see Theorem 5.3).

Let A be a space and $B \subseteq A$. In this paper, the notation A/B stands for a quotient space of A after shrinking B to a single point. The normality of a space is needed to assume this space to be completely regular.

2. Notation and terminology

Recall that a space X is of *countable tightness* if whenever $A \subseteq X$ and $x \in \overline{A}$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$. A space X is a *k-space* provided that a subset $C \subseteq X$ is closed in X if $C \cap K$ is closed in K for each compact subset K of X. A space X is a *sequential space* if for any non-closed set A of X there is a sequence in A such that this sequence converges to some point in $X \setminus A$. A space X is called a *k_ω-space* if $X = \bigcup_{i\in\omega} X_i$, where each X_i is compact, and each set $E \subseteq X$ such that every $E \cap X_i$ closed in X_i is closed in X. It is known that [23]

(1) every first-countable space is a sequential space;
(2) every sequential space is of countable tightness and a k-space;
(3) every k-space with compact subsets metrizable is a sequential space;
(4) every hereditarily separable space is of countable tightness;
(5) every Lindelöf locally k_ω-space is a k_ω-space;
(6) every k_ω-space is a σ-compact, k-space.

Definition 2.1. Let $\mathcal{P}$ be a cover of a space X.

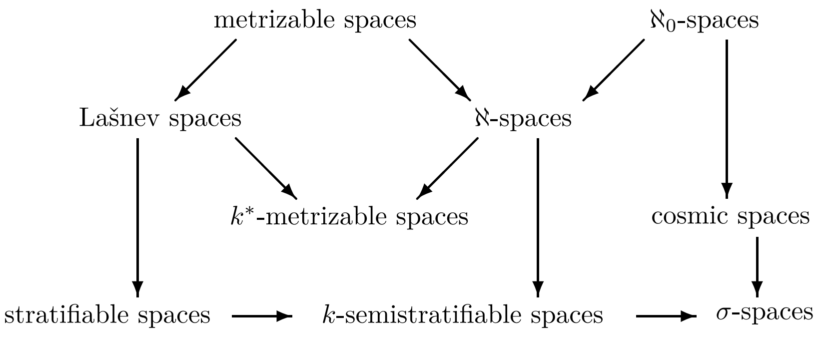
- The family $\mathcal{P}$ is a *network* of X [1] if for each U open in X and $x \in U$ there is a $P \in \mathcal{P}$ such that $x \in P \subseteq U$;
- $\mathcal{P}$ is called a *k-network* of X [24] if for each U open in X and each compact set $K \subseteq U$, there is a finite subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \bigcup \mathcal{P}' \subseteq U$.

A regular space X is called a *σ-space* (resp. an *$\aleph$-space* [24]) if it has a σ-locally finite network (resp. a σ-locally finite k-network). A regular space X with a countable network (resp. a countable k-network) is called a *cosmic space* (resp. an *$\aleph_0$-space* [22]). Every compact subset of σ-spaces is metrizable [9]. Every cosmic space is hereditarily separable and hereditarily Lindelöf.

We shall concern ourselves with two classes of generalized metrizable spaces: k^*-metrizable spaces and k-semistratifiable spaces. *k^*-Metrizable spaces* [4] are defined by certain images of metric spaces, which can be characterized as regular spaces with a σ-compact-finite k-network (see [4, Theorem 6.4]), in which a family $\mathscr{P}$ of subsets of a space X is *compact-finite* if every compact subset of X meets at most finitely many $P \in \mathscr{P}$. A regular space X is said to be a *k-semistratifiable space* [19], if there is an operator U assigning to each closed set F, a sequence of open sets $U(F) = \{U(n, F)\}_{n\in\mathbb{N}}$ such that (1) $\bigcap_{n\in\mathbb{N}} U(n, F) = F$; (2) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$; (3) if K is compact in X and $K \cap F = \varnothing$, then $K \cap U(m, F) = \varnothing$ for some $m \in \mathbb{N}$. If we replace the above condition (1) and (3) by (1′) $\bigcap_{n\in\mathbb{N}} U(n, F) = \bigcap_{n\in\mathbb{N}} \overline{U(n, F)} = F$, then X is said to be a *stratifiable space* [5].

A space X is called a *Lašnev space* if X is a closed image of a metric space. Lašnev spaces can be characterized as a regular Fréchet space with a σ-compact-finite k-network [16], where a space X is called *a Fréchet space* if, for any $A \subseteq X$ and $x \in \overline{A}$, there is a sequence $S \subseteq A$ converging to x.

We summarize some relations of above generalized metric spaces as follows [4,9,14].



The free topological group $F(X)$ admits the following description: there is a canonical mapping: $i : X \to F(X)$, and for any continuous mapping f of X to a topological group G there exists a unique continuous homomorphism $h : F(X) \to G$ such that $f = h \circ i$. It is known that X is a Tychonoff space if and only if the canonical mapping i is a closed embedded mapping into $F(X)$. Therefore, the topology of the group $F(X)$ over a Tychonoff space can be defined as the strongest group topology that induces the original topology on X. Regarded as a group without a topology, $F(X)$ is the free group of the set X. Abelian topological group $A(X)$ is defined similarly. Given a subset Y of a Tychonoff space X, we use $F(Y, X)$ to denote the subgroup of $F(X)$ generated by Y, while $A(Y, X)$ stands for the corresponding subgroup of $A(X)$. It is well known that if Y is closed in a Tychonoff space X, then $F(Y, X)$ and $A(Y, X)$ are closed in $F(X)$ and $A(X)$, respectively.

For a space X, we always denote the set of all isolated points of X by $I(X)$ and $NI(X) = X \setminus I(X)$ in this paper. We denote by $\mathbb{N}$ the set of all natural numbers. Readers may refer [3,6,9] for notations and terminology not explicitly given here. In what follows, all spaces are assumed to be Tychonoff, which means that all spaces are completely regular and Hausdorff.

3. Some basic facts

In this section, we shall establish some basic facts, which play an important role in Sections 4 and 5.

Recall that a space X is *ω_1-compact* if every closed discrete subset of X is countable. A space $(X, \mathscr{T})$, where $\mathscr{T}$ is the topology of X, is *submetrizable* if, there is a metrizable topology $\mathscr{T}'$ on X such that $\mathscr{T}' \subseteq \mathscr{T}$. Every cosmic space is submetrizable [5].

Proposition 3.1. Let X be a sequential space such that $NI(X)$ is ω_1-compact and submetrizable. If $F(X)$ is either of countable tightness or a k-space, then X is ω_1-compact or discrete.

Proof. Suppose that the space X is neither ω_1-compact nor discrete. Then there is an uncountable discrete closed subset D in X. Since $NI(X)$ is ω_1-compact and closed in X, we can assume that $D \subseteq I(X)$.

Case 1. $I(X)$ is not closed in X.

From the fact that X is a sequential space, it follows that there is a non-trivial sequence $S \subseteq I(X)$ converging to some point $x \in X \setminus I(X)$. Without loss of generality, we can assume that $S \cap D = \varnothing$. Put $F = X \setminus (D \cup S)$ and $Z = X/F$. Observing that every point in $S \cup D$ is an isolated point in X and D is both closed and open in X, one can easily check that the space Z is homeomorphic to $D \oplus \overline{S}$. Then $F(Z)$ is neither of countable tightness nor a k-space [2, Proposition 3.2]. One the other hand, let $p : X \to Z$ be the natural quotient mapping, where Z is metrizable. The homomorphism $\widetilde{p} : F(X) \to F(Z)$ extending the quotient mapping p is open by [3, Corollary 7.1.9]. Since the countable tightness and k-property are preserved by quotient mappings [23], $F(Z)$ is of either countable tightness or a k-space, which is a contradiction.

Case 2. $I(X)$ is closed in X.

Since $NI(X)$ is submetrizable, that is, there is a one-to-one continuous mapping f from $NI(X)$ onto a metrizable space M. Define $\widetilde{f} : X \to Z = M \oplus I(X)$ as follows: $\widetilde{f}(x) = f(x)$ when $x \in NI(X)$, otherwise, $\widetilde{f}(x) = x$. Note that $I(X)$ is closed and open in X, so $\widetilde{f}(x)$ is continuous. Observing that X is a non-discrete sequential space and $NI(X)$ is closed and open, the subspace $NI(X)$ is both non-discrete and sequential, and therefore, there is a non-trivial sequence S converging to some point x in $NI(X)$. Clearly, the sequence $\widetilde{f}(S)$ converges to $\widetilde{f}(x)$ in Z and $\widetilde{f}(D)$ is an uncountable closed and discrete subset in Z. One can easily show that $\overline{S} \cup D$, $\widetilde{f}(\overline{S}) \cup \widetilde{f}(D)$ and $\overline{S} \oplus D$ are homeomorphic.

The homomorphism $F(\widetilde{f}) : F(X) \to F(Z)$ extending the continuous mapping $\widetilde{f}$ is continuous by [3, Corollary 7.1.9]. Observing that the mapping $\widetilde{f}$ is both one-to-one and continuous, the mapping $F(\widetilde{f})$ is both one-to-one and continuous as well, so is the mapping $F(\widetilde{f})_{|F(\overline{S} \cup D, X)} : F(\overline{S} \cup D, X) \to F(\widetilde{f}(\overline{S}) \cup \widetilde{f}(D), Z)$.

Next, we shall show that $F(\widetilde{f})_{|F(\overline{S}\cup D,X)}$ is homeomorphism. Since Z is metrizable, according to [26, Theorem 1] the free topological group $F(\widetilde{f}(\overline{S}) \oplus \widetilde{f}(D))$ is topologically isomorphic to $F(\widetilde{f}(\overline{S}) \cup \widetilde{f}(D), Z)$, i.e, $F(\widetilde{f}(\overline{S}) \oplus \widetilde{f}(D)) \cong F(\widetilde{f}(\overline{S}) \cup \widetilde{f}(D), Z)$. Thus, from the fact that the topology of the group $F(T)$ is the finest topological group topology on $F_a(T)$ that generates on T its original topology [3, Corollary 7.1.8] it follows that

$$F(\overline{S} \cup D, X) \cong F(\widetilde{f}(\overline{S}) \oplus \widetilde{f}(D)) \cong F(\widetilde{f}(\overline{S}) \cup \widetilde{f}(D), Z).$$

Since $\overline{S} \cup D$ is closed in X, the set $F(\overline{S} \cup D, X)$ is closed in $F(X)$ by [3, Theorem 7.4.5]. Hence, we have proved that there is a closed copy of $F(\widetilde{f}(\overline{S}) \oplus \widetilde{f}(D))$ in $F(X)$. Thus, $F(X)$ is neither of countable tightness nor a k-space because $F(\widetilde{f}(\overline{S}) \oplus \widetilde{f}(D))$ is neither of countable tightness nor a k-space [2, Proposition 3.2]. This is a contradiction.

Hence, X is ω_1-compact or discrete. □

A family $\mathscr{P}$ of subsets of a space X is *cs-finite* if every convergent sequence of X meets at most finitely many $P \in \mathscr{P}$.

Lemma 3.2. *Let $\mathscr{P}$ be a cs-finite family of subsets in a sequential space X. Then the set $\bigcup\{s_P : P \in \mathscr{P}\}$ is closed in X, where s_P is a finite subset of P; moreover, $\mathscr{P}$ is countable if X is ω_1-compact.*

Proof. Choose a finite set $s_P \subseteq P$ for each $P \in \mathscr{P}$. Without loss of generality, we can assume that $A = \bigcup\{s_P : P \in \mathscr{P}\}$ is infinite. If the subset A is not closed in X, there is a non-trivial sequence S in A converging to $x \in X \setminus A$. It is obvious that the family $\{P \in \mathscr{P} : P \cap S \neq \varnothing\}$ is infinite, and $\mathscr{P}$ is not *cs*-finite. This is a contradiction.

If $\mathscr{P}$ is uncountable, there is an uncountable set $A = \{x_\alpha : \alpha < \omega_1\}$ in X and an uncountable subfamily $\{P_\alpha \in \mathscr{P} : \alpha < \omega_1\}$ such that $x_\alpha \in P_\alpha$ for each $\alpha < \omega_1$. Then A is closed discrete in X, and X is not ω_1-compact. □

Recall that the space S_{ω_1} is the quotient space obtained by identifying all the limit points of the topological sum of ω_1 many nontrivial convergent sequences.

Lemma 3.3. *If $F(X)$ is either of countable tightness or a k-space, then every quotient Y of X, where Y is Tychonoff, contains no closed copy of S_{ω_1}.*

Proof. Let $p : X \to Y$ be a quotient mapping, where Y is also Tychonoff. The homomorphism $F(p) : F(X) \to F(Y)$ extending the quotient mapping p is open by [3, Corollary 7.1.9]. Since the countable tightness and k-space property are preserved by quotient mappings [23], $F(Y)$ is either of countable tightness or a k-space. Observe that the space $S_{\omega_1} \times S_{\omega_1}$ is neither of countable tightness nor a k-space [8] and that the free topological group $F(Y)$ contains a closed copy of $Y \times Y$ [3, Theorem 7.1.13], Y has no closed copy of S_{ω_1}. □

A space X is *collectionwise Hausdorff* if, for any discrete set $\{x_\alpha : \alpha \in \Gamma\} \subseteq X$, there is a pairwise disjoint family $\{U_\alpha : \alpha \in \Gamma\}$ of open sets in X such that $x_\alpha \in U_\alpha$ for each $\alpha \in \Gamma$.

Proposition 3.4. Let X be a sequential space. If $F(X)$ is either of countable tightness or a k-space, then X is cosmic or discrete if and only if X is a normal collectionwise Hausdorff space with a σ-*cs*-finite network.

Proof. Obviously, every cosmic space or every discrete space is a normal collectionwise Hausdorff space with a σ-*cs*-finite network.

Conversely, suppose that X is a normal collectionwise Hausdorff space with a σ-*cs*-finite network. It is enough to show that X is ω_1-compact or discrete by Lemma 3.2.

Firstly, we show that the subspace $NI(X)$ is ω_1-compact. If not, choose a closed uncountable discrete subset $D = \{x_\alpha : \alpha < \omega_1\}$ in $NI(X)$. Observing that X is normal collectionwise Hausdorff, it is easy to check there is a discrete collection $\{U_\alpha : \alpha < \omega_1\}$ of open sets in X such that $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$. Since X is sequential and $D \subseteq NI(X)$ is a closed discrete subset, we can find a non-trivial sequence $\{x_\alpha(n)\} \subseteq U_\alpha \setminus D$ converging to x_α for each $\alpha < \omega_1$. Put $Y = X/D$. Then the quotient mapping $p : X \to Y$ is closed and Y is normal. Let $E = \{x_\alpha(n) : n < \omega, \alpha < \omega_1\} \cup D$. One can easily check that $p(E)$ is a closed copy of S_{ω_1} in Y. This is a contradiction with Lemma 3.3, thus $NI(X)$ is ω_1-compact.

It is obvious that $NI(X)$ as a closed subspace of X is a sequential space with a σ-*cs*-finite network, so that from Lemma 3.2 it follows that $NI(X)$ has a countable network, and therefore, $NI(X)$ is submetrizable. Hence, the statement directly follows from Proposition 3.1. □

A family $\mathcal{P}$ of subsets of a space X is called a *wcs*-network* of X [15] if whenever a sequence $\{x_n\}$ converges to $x \in U$, where U is open in X, there is a $P \in \mathcal{P}$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i} : i \in \mathbb{N}\} \subseteq P \subseteq U$. Obviously, k-networks are *wcs**-networks, and *wcs**-networks are networks in a space. If we strengthen the network to a *wcs**-network in Proposition 3.4, then the collectionwise Hausdorff property can be dropped as follows.

Proposition 3.5. Let X be a sequential space. If $F(X)$ is either of countable tightness or a k-space, then X is an $\aleph_0$-space or discrete if and only if X is a normal space with a σ-*cs*-finite *wcs**-network.

Proof. It is enough to prove the sufficiency. Let X be a normal space with a σ-*cs*-finite *wcs**-network. It is known that a Hausdorff space Y with a countable *wcs**-network has a countable k-network [10], we only need to show that $NI(X)$ is ω_1-compact by Lemma 3.2 and the proof of Proposition 3.4.

Suppose that $NI(X)$ is non-ω_1-compact. Choose a closed discrete uncountable subset $D = \{x_\alpha : \alpha < \omega_1\}$ in $NI(X)$. Then for each $\alpha < \omega_1$, choose an open set V_α such that $x_\alpha \in V_\alpha \subseteq \overline{V_\alpha} \subseteq X \setminus \{x_\beta : \beta < \omega_1, \beta \neq \alpha\}$. Observing that X is a sequential space, one can find a non-trivial sequence $\{x_\alpha(n)\} \subseteq V_\alpha$ of X converging to x_α. Let $\mathscr{P} = \bigcup_{n\in\omega} \mathscr{P}_n$ be a σ-*cs*-finite *wcs**-network of X, where each $\mathscr{P}_n$ is *cs*-finite. There is a $P_\alpha \in \mathscr{P}$ and a subsequence $\{x_\alpha(n_i)\}$ of $\{x_\alpha(n)\}$ such that $\{x_{\alpha(n_i)}\} \subseteq P_\alpha \subseteq V_\alpha$.

Without loss of generality, we may assume the collection $\{P_\alpha : \alpha < \omega_1\} \subseteq \mathscr{P}_m$ for some $m \in \omega$. Since $\mathscr{P}_m$ is *cs*-finite, we may further assume that $\{x_\alpha(n_i) : i \in \omega\} \cap \{x_\beta(n_i) : i \in \omega\} = \varnothing$ whenever $\alpha \neq \beta$. Put $E = \{x_\alpha(n_i) : i \in \omega, \alpha < \omega_1\} \cup D$ and $Y = X/D$. Then the quotient mapping $p : X \to Y$ is closed and Y is normal. Applying Lemma 3.2 one can easily check that $p(E)$ is a copy of S_{ω_1}. To show $p(E)$ is closed in Y, it is enough to show that E is closed in X. If not, there is a non-trivial sequence $l \subseteq E \setminus D$ converging to some point $y \in X \setminus E$. Noting that $y \notin D$, the set $l \cap \{x_\alpha(n_i) : i < \omega\}$ is finite for each $\alpha < \omega_1$. Hence, $\{P \in \mathscr{P}_m : l \cap P \neq \varnothing\}$ is infinite, a contradiction because $\mathscr{P}_m$ is *cs*-finite. Therefore, E is closed in X, and $p(E)$ is a closed copy of S_{ω_1} in Y. It is a contradiction with Lemma 3.3. □

Remark 3.6. In view of the proof of Proposition 3.5, the quotient space Y is Tychonoff if X is normal. It will be shown in Theorem 5.3 that the normality of X can be omitted when $F(X)$ is a k-space. We don't known whether Proposition 3.5 is true if drop the normality of X when $F(X)$ is of countable tightness.

4. The countable tightness of free topological groups

In this section, we shall discuss the countability of tightness of free topological groups on some generalized metric spaces. Some characterizations are given. The first main result is as follows, which improves Theorems 1.3 and 1.5.

Theorem 4.1. *Let X be a paracompact σ-, k-space. Then the following statements are equivalent:*

(1) *$F(X)$ is of countable tightness;*
(2) *X is cosmic or discrete;*
(3) *X is separable or discrete.*

Proof. (1) ⇒ (3). Suppose that X is a paracompact σ-, k-space. Then X is a normal collectionwise Hausdorff sequential space with a σ-locally finite network. Thus the implication immediately follows from Proposition 3.4.

The implication (3) ⇒ (2) being obvious, we pass to the proof that (2) ⇒ (1). If X is cosmic or discrete, then so is $F(X)$ [3, Corollary 7.1.17], and hence, the tightness of $F(X)$ is countable. □

Every normal k-semistratifiable k-space is paracompact [13, Theorem 2].

Corollary 4.2. *Let X be a normal k-semistratifiable k-space. Then the following statements are equivalent:*

(1) *$F(X)$ is of countable tightness;*
(2) *X is cosmic or discrete;*
(3) *X is separable or discrete.*

Next, we shall consider the countability of tightness in free topological groups on k^*-metrizable spaces.

Recently, Z. Li et al. [11, Theorem 4.2] proved that: Let X be a k^*-metrizable k-space. If $I(X) = \varnothing$, then $F(X)$ is of countable tightness if and only if X is an $\aleph_0$-space. However, we find that Z. Li et al didn't consider the separation axiom of the quotient space Y in the proof of [11, Lemma 4.1, p. 192, line 20]. Thus its proof seems to have a gap.

Since every k^*-metrizable k-space is sequential [4, Theorem 3.5], we have the following results by Theorem 4.1 and Proposition 3.5.

Theorem 4.3. *Let X be a normal k^*-metrizable k-space. Then $F(X)$ is of countable tightness if and only if X is an $\aleph_0$-space or discrete.*

Corollary 4.4. *Let X be a Lašnev space. Then the following statements are equivalent:*

(1) *$F(X)$ is of countable tightness;*
(2) *X is an $\aleph_0$-space or discrete;*
(3) *X is separable or discrete.*

Proof. Since every Lašnev space is a paracompact k^*-metrizable k-space [16], it is enough to show that (3) ⇒ (2) by Theorem 4.3. Suppose that X is a separable Lašnev space. Then X is a Fréchet Lindelöf space with a σ-compact-finite k-network, thus X has a countable k-network by Lemma 3.2, i.e., X is an $\aleph_0$-space. □

Assuming the continuum hypothesis (CH) every separable k^*-metrizable k-space is an $\aleph_0$-space by [17, Theorem 7].

Corollary 4.5 *(CH). Let X be a normal k^*-metrizable k-space. Then $F(X)$ is of countable tightness if and only if X is separable or discrete.*

In view of Remark 3.6, we have the following a question.

Question 4.6. *Can the conditions normality in Corollary 4.2 and Theorem 4.3 be dropped?*

5. *k*-Property in free topological groups

In this section, we mainly consider the k-space property of free topological groups on σ-spaces and k^*-metrizable spaces.

Lemma 5.1 *([3, Exercises 7.6.f]). If X is a first-countable space and $A(X)$ is a k-space, then X is locally pseudocompact.*

The following result improves Theorem 1.4.

Theorem 5.2. *Let X be a first-countable paracompact σ-space. Then the following conditions are equivalent:*

(1) *$F(X)$ is a k-space;*
(2) *$F(X)$ is a k_ω-space or discrete;*
(3) *X is locally compact separable metrizable or discrete.*

Proof. The implication (2) ⇒ (1) is obvious.

(1) ⇒ (3). Without loss of generality we can assume that X is non-discrete. Observe that $F(X)$ is a k-space, so is $A(X)$ because $A(X)$ is a continuous open image of $F(X)$ [3, Theorem 7.1.11], and, therefore, X is locally pseudocompact by Lemma 5.1. By Proposition 3.4 we obtain that X is cosmic. Then X is locally compact, and therefore X is also a separable metrizable space [22].

(3) ⇒ (2). It is well known that every a locally compact separable metrizable space is a k_ω-space, so the implication (3) ⇒ (2) follows from Theorem 1.1. □

Theorem 1.6 was showed by [11, Lemma 4.1], and the quotient space Y (see [11, Lemma 4.1, p. 192, line 20]) may not be Tychonoff, so the proof of Theorem 1.6 seems to have a gap. The following Theorem 5.3 is a better result.

Let $\mathscr{P}$ be a family of subsets of a space X. $\mathscr{P}$ is *compact-countable* if every compact subset of X meets at most countably many $P \in \mathscr{P}$. $\mathscr{P}$ is *star-countable* if every $P \in \mathscr{P}$ meets at most countably many other $Q \in \mathscr{P}$.

Theorem 5.3. *Let X be a k^*-metrizable space. Then the following conditions are equivalent:*

(1) *$F(X)$ is a k-space;*
(2) *$F(X)$ is a k_ω-space or is discrete;*
(3) *X is a k_ω-space or discrete;*
(4) *X is locally k_ω separable or discrete.*

Proof. It is enough to show the implications (1) ⇒ (4) ⇒ (3).

(1) ⇒ (4). Suppose that $F(X)$ is a k-space.

Since X is a k^*-metrizable space, X has a compact-countable k-network. In addition, X^2 is a k-space, so that X is either first-countable or locally k_ω by [18, Theorem 3.4].

Case 1. X is first-countable.

Since X is k^*-metrizable, X is metrizable by [16, Corollary 1]. Then X is locally compact separable or discrete by Theorem 5.2.

Case 2. X is locally k_ω.

Since X has a σ-compact-finite k-network, X has a compact-countable k-network $\mathscr{P}$ such that the set $\overline{P}$ is σ-compact for each $P \in \mathscr{P}$. Hence, $\mathscr{P}$ is a star-countable k-network of X, and X is a paracompact σ-space by [25, Corollary 2.4]. By Proposition 3.5, X is separable or discrete.

(4) $\Rightarrow$ (3). Suppose that X is locally k_ω and separable. By the case 2, X is paracompact, thus X is Lindelöf and locally k_ω, hence X is a k_ω-space. □

However, we have the following a question.

Question 5.4. *Does Theorem 5.3 hold for normal k-semistratifiable spaces?*

Acknowledgement

We wish to thank the referee for the detailed list of corrections, suggestions and all her/his efforts in order to improve the paper.

References

[1] A.V. Arhangel'skiĭ, An addition theorem for the weight of sets lying in bicompacts, (in Russian) Dokl. Akad. Nauk SSSR 126 (2) (1959) 239–241.
[2] A.V. Arhangel'skiĭ, O.G. Okunev, V.G. Pestov, Free topological groups over metrizable spaces, Topol. Appl. 33 (1989) 63–76.
[3] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, World Sci., 2008.
[4] T. Banakh, V. Bogachev, A. Kolesnikov, k^*-Metrizable spaces and their applications, J. Math. Sci. 155 (4) (2008) 475–522.
[5] C.R. Borges, On stratifiable spaces, Pac. J. Math. 17 (1966) 1–16.
[6] R. Engelking, General Topology, revised and completed edition, Heldermann Verlag, 1989.
[7] T. Fay, E. Ordman, B.V.S. Thomas, The free topological groups over rationals, Gen. Topol. Appl. l0 (1979) 33–47.
[8] G. Gruenhage, Y. Tanaka, Products of k-spaces and spaces of countable tightness, Trans. Am. Math. Soc. 273 (1982) 299–308.
[9] G. Gruenhage, Generalized metric spaces, in: K. Kunen, E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.
[10] J.A. Guthrie, A characterization of $\aleph_0$-spaces, Gen. Topol. Appl. 1 (1971) 105–110.
[11] Z. Li, F. Lin, C. Liu, Networks on free topological groups, Topol. Appl. 180 (2015) 186–198.
[12] F. Lin, C. Liu, S_ω and S_2 on free topological groups, Topol. Appl. 176 (2014) 10–21.
[13] S. Lin, Covering properties of k-semistratifiable spaces, Topol. Proc. 29 (2005) 199–206.
[14] S. Lin, Generalized Metric Spaces and Mappings, second edition, China Science Press, Beijing, 2007.
[15] S. Lin, Y. Tanaka, Point-countable k-networks, closed maps, and related results, Topol. Appl. 59 (1994) 79–86.
[16] C. Liu, Spaces with a σ-compact finite k-network, Quest. Answ. Gen. Topol. 10 (1992) 81–87.
[17] C. Liu, Y. Tanaka, Spaces having σ-compact-finite k-networks, and related matters, Topol. Proc. 21 (1996) 173–200.
[18] C. Liu, Y. Tanaka, Star-countable k-networks, compact-countable k-networks, and related results, Houst. J. Math. 24 (4) (1998) 655–670.
[19] D.J. Lutzer, Semimetrizable and stratifiable spaces, Gen. Topol. Appl. 1 (1971) 43–48.
[20] J. Mack, S.A. Morris, E.T. Ordman, Free topological groups and the projective dimension of locally compact Abelian groups, Proc. Am. Math. Soc. 40 (1) (1973) 303–308.
[21] A.A. Markov, On free topological groups, Izv. Akad. Nauk SSSR, Ser. Mat. 9 (1945) 3–64.
[22] E.A. Michael, $\aleph_0$-Spaces, J. Math. Mech. 15 (1966) 983–1002.
[23] E.A. Michael, A quintuple quotient quest, Gen. Topol. Appl. 2 (1972) 91–138.
[24] P. O'Meara, On paracompactness in function spaces with the compact open topology, Proc. Am. Math. Soc. 29 (1971) 183–189.
[25] M. Sakai, On space with a star-countable k-network, Houst. J. Math. 23 (1) (1997) 45–56.
[26] V. Uspenskii, Free topological groups of metrizable spaces, Math. USSR, Izv. 37 (1991) 657–680.
[27] K. Yamada, Metrizable subspaces of free topological groups on metrizable spaces, Topol. Proc. 23 (1998) 379–409.
[28] K. Yamada, Fréchet–Urysohn spaces in free topological groups, Proc. Am. Math. Soc. 130 (8) (2002) 2461–2469.