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S_2 and the Fréchet property of free topological groups ☆

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ABSTRACT

Let $F(X)$ denote the free topological group over a Tychonoff space X, $F_n(X)$ denote the subspace of $F(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X for every non-negative integer n and $E_n(X) = F_n(X) \setminus F_{n-1}(X)$ for $n \geq 1$. In this paper, we study topological properties of free topological groups in terms of Arens' space S_2. The following results are obtained.
(1) If the free topological group $F(X)$ over a Tychonoff space X contains a non-trivial convergent sequence, then $F(X)$ contains a closed copy of S_2, equivalently, $F(X)$ contains a closed copy of S_ω, which extends [6, Theorem 1.6].
(2) Let X be a topological space and $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is κ-Fréchet–Urysohn and contains no copy of S_2, then X is discrete, which improves [15, Proposition 3.5].
(3) If X is a μ-space and $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete, which improves [15, Theorem 2.4].
At last, a question posed by K. Yamada is partially answered in a shorter alternative way by means of a Tanaka's theorem concerning Arens' space S_2.

1. Introduction

In 1941, the free topological group $F(X)$ over a Tychonoff space X in the sense of Markov was introduced [9]. Topologists discussed various topological properties on free topological groups, where sequentiality and the Fréchet property, as important topological properties, were investigated.

In 2014, F. Lin, C. Liu [6] showed the following.

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Theorem 1.1. *([6, Theorem 1.6]) If the free topological group $F(X)$ over a Tychonoff space X is a sequential space, then either X is discrete or $F(X)$ contains a copy of S_ω.*

In 2002, K. Yamada [15] investigated the Fréchet property of the subspace $F_n(X)$ of the free topological group $F(X)$ over a metrizable space X, where $F_n(X)$ denotes the subspace of $F(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X for every non-negative integer n, and obtained the following results.

Theorem 1.2. *([15, Corollary 2.5]) Let X be a metrizable space. $F_3(X)$ is Fréchet–Urysohn if and only if the set of all non-isolated points of X is compact.*

Theorem 1.3. *([15, Theorem 2.4]) Let X be a metrizable space. If $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete.*

K. Yamada [15] posed following Question 1.4 and conjectured that the answer to this question is affirmative.

Question 1.4. Let X be a metrizable space. Is $F_4(X)$ is Fréchet–Urysohn if the set of all non-isolated points of X is compact?

F. Lin and C. Liu [6] tried to solve Question 1.4, however, there was a gap in the proof [6,7]. Hence Question 1.4 is still open.

In this paper, we shall make full use of the concept of Arens' space S_2 to establish our main results. This paper is organized as follows.

At first, we shall extend Theorem 1.1 by proving that if the free topological group $F(X)$ over a Tychonoff space X contains a non-trivial convergent sequence, then $F(X)$ contains a closed copy of S_2, equivalently, $F(X)$ contains a closed copy of S_ω.

Secondly, let X be a topological space and $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is κ-Fréchet–Urysohn and contains no copy of S_2, then X is discrete, which improves [15, Proposition 3.5].

Thirdly, we shall prove that if X is a μ-space and $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete, which improves Theorem 1.3.

Quite recently, K. Yamada [16], in a lengthy proof, proved that if X is a locally compact, metrizable space and the set of all non-isolated points of X is compact, then $F_4(X)$ is a k-space if and only if $F_4(X)$ is a Fréchet–Urysohn space. Further, if X is a locally compact, separable, metrizable space, then the set of all non-isolated points of X is compact if and only if $F_4(X)$ is a Fréchet–Urysohn space, which gave a partial answer to Question 1.4. In this paper, we shall present a shorter alternative way to prove the above result by means of a Tanaka's theorem concerning Arens' space S_2.

2. Preliminaries

A topological space X is called a *Fréchet* space or *Fréchet–Urysohn* (*κ-Fréchet–Urysohn*) space if for every $A \subset X$ (open subset $A \subset X$) and every $x \in \overline{A}$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of A converging to x. A topological space X is called a *sequential* space if a set $A \subset X$ is closed if and only if together with any sequence it contains its limits. Obviously, every Fréchet–Urysohn space is sequential (κ-Fréchet–Urysohn). A κ-Fréchet–Urysohn space need not be sequential [8].

Definition 2.1. ([1]) Let $X = \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$. $\mathbb{N}^\mathbb{N}$ denotes the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. For every $n, m, k \in \mathbb{N}$, put $V(n, m) = \{n\} \cup \{(n, k) : k \geq m\}$. For every $x \in \mathbb{N}^2$, let $\mathcal{B}(x) = \{\{x\}\}$. For every $n \in \mathbb{N}$, let $\mathcal{B}(n) = \{V(n, m) : m \in \mathbb{N}\}$. Let $\mathcal{B}(0) = \{\{0\} \cup \bigcup_{n\geq i} V(n, f(n)) : i \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N}\}$. The topological space X, generated by the neighborhood system $\{\mathcal{B}(x)\}_{x\in X}$, is called *Arens'* space and denoted by S_2.

Obviously, the subspace $Y = \{0\} \cup \{n_i : i \in \mathbb{N}\} \cup \{(n_i, m_j(i)) : i, j \in \mathbb{N}\}$ of S_2 is homeomorphic to S_2, where $\{n_i\}_{i\in\mathbb{N}}$ is an arbitrary sequence with $n_1 < n_2 < \cdots$ and $\{m_j(i)\}_{j\in\mathbb{N}}$ is an arbitrary sequence with $m_1(i) < m_2(i) < \cdots$.

It is easy to see that Arens' space S_2 is sequential but not κ-Fréchet–Urysohn.

A topological space X is called a *k-space* [4] if for every $A \subset X$, the set A is closed in X provided that the intersection of A with any compact subspace Z of the space X is closed in Z. The following Tanaka's theorem was established in 1983.

Theorem 2.2. *([13]) Suppose that X is a k-space in which every singleton is a G_δ-set. Then X is Fréchet–Urysohn if X contains no closed copy of S_2.*

In the paper, $F_a(X)$ algebraically denotes the free group on non-empty set X and e is the identity of $F_a(X)$. The set X is called a free basis of $F_a(X)$. Here are some details. Every $g \in F_a(X)$ distinct from e has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, ..., x_n \in X$ and $\varepsilon_1, ..., \varepsilon_n = \pm 1$. This expression or word for g is called reduced if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$ and we say that the length $l(g)$ of g equals to n. Every element $g \in F_a(X)$ distinct from the identity e can be uniquely written in the form $g = x_1^{r_1} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$ and $x_i \neq x_{i+1}$ for every $i = 1, ..., n-1$.

Remark 2.3. It has been shown, for instance, see [2], that the topology of the free topological group $F(X)$ over a Tychonoff space X is the finest topological group topology on the group $F_a(X)$ which induces the original topology on X.

For every non-negative integer n, $F_n(X)$ denotes the subspace of $F(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X. Put $E_n(X) = F_n(X) \setminus F_{n-1}(X)$ for $n \geq 1$. The symbol i_n denotes the continuous multiplication mapping of $\tilde{X}^n$ onto $F_n(X)$ for every $n \in \mathbb{N}$, where $\tilde{X} = X \oplus \{e\} \oplus X^{-1}$.

Lemma 2.4. *([2, Theorem 7.1.13]) Let X be a topological space. The subspaces X and $F_n(X)$ of $F(X)$ are all closed in $F(X)$ for every non-negative integer n.*

Lemma 2.5. *([2, Corollary 7.4.3]) Let X be a topological space and C be any set of $F(X)$. If $C \cap F_n(X)$ is finite for every $n \in \mathbb{N}$, then C is closed and discrete in $F(X)$.*

Lemma 2.6. *([2, Corollary 7.4.4]) Let X be a topological space and K be a countably compact subspace of $F(X)$. Then $K \subset F_n(X)$ for some $n \in \mathbb{N}$.*

Lemma 2.7. *([2, Corollary 7.4.6]) If C is a compact subset of a topological space X, then $F(C, X)$ is topologically isomorphic to $F(C)$, where $F(C, X)$ is the subgroup of $F(X)$ generated by C.*

The *support* [2] of a reduced word $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in F(X)$, where $x_1, ..., x_n \in X$ and $\varepsilon_1, ..., \varepsilon_n = \pm 1$, is defined as follows:

$$\operatorname{supp}(g) = \{x_1, ..., x_n\}.$$

Given a subset K of $F(X)$, we put

$$\operatorname{supp}(K) = \bigcup_{g \in K} \operatorname{supp}(g).$$

A subset B of a topological space X is said to be bounded in X (or simply bounded) if every continuous real-valued function on X is bounded on B [2]. A topological space X is called a μ-space, if the closure of every bounded set in X is compact [2]. It is easy to see that every paracompact space is a μ-space.

Lemma 2.8. *([2, Corollary 7.5.6]) Let X be a μ-space. If K is a bounded subset of $F(X)$, then the closure of supp(K) in X is compact.*

In what follows, all topological spaces are assumed to be Tychonoff, unless stated otherwise. For some terminology unstated here, readers may refer to [2,4].

3. The Fréchet property and S_2 of $F(X)$

At first, we shall improve Theorem 1.1 in the introduction. We need a technical lemma.

Lemma 3.1. *Let X be a topological space and $L = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of $F(X) \setminus \{e\}$ converging to the identity e. For every $p \in \mathbb{N}$, there exist $q \in \mathbb{N}, y \in X \cup X^{-1}$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{y^q x_{n_k} y^{-q}\}_{k\in\mathbb{N}}$ converges to e, and $l(y^q x_{n_k} y^{-q}) > p$ for every $k \in \mathbb{N}$.*

Proof. By Lemma 2.6, $L \subset F_m(X)$ for some $m \in \mathbb{N}$. Without loss of generality, we assume, for every $n \in \mathbb{N}$, $l(x_n) = s$ for some $s \leq m$. We write $x_n = x_{n,1} \cdots x_{n,s}$ for every $n \in \mathbb{N}$, where $x_{n,1}, ..., x_{n,s} \in X \cup X^{-1}$. It is easy to see that either $|\{n : x_{n,1} \in X\}| = \omega$ or $|\{n : x_{n,1} \in X^{-1}\}| = \omega$.

Without loss of generality, we may assume $x_{n,1} \in X$ for every $n \in \mathbb{N}$. If $|\{n : x_{n,s} = x_0, n \in \mathbb{N}\}| = \omega$ for some $x_0 \in X \cup X^{-1}$, we pick $y \in X$ such that $y \neq x_0$ if $x_0 \in X$; and $y = x_0^{-1}$ if $x_0 \in X^{-1}$. Let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of L with $x_{n_k,s} = x_0$ for every $k \in \mathbb{N}$. Choose $q > p$, then $\{y^q x_{n_k} y^{-q}\}_{k\in\mathbb{N}}$ converges to e and $l(y^q x_{n_k} y^{-q}) = 2q + s > p$ for every $k \in \mathbb{N}$. Otherwise, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of L such that $x_{n_i,s} \neq x_{n_j,s}$ if $i \neq j$. Let $y = x_{n_1,s}$ if $x_{n_1,s} \in X$, and $y = x_{n_1,s}^{-1}$ if $x_{n_1,s} \in X^{-1}$. Choose $q > p$, then there exists a subsequence $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\{y^q x_{n_{k_j}} y^{-q}\}_{k\in\mathbb{N}}$ converges to e and $l(y^q x_{n_{k_j}} y^{-q}) = 2q + s > p$ for every $j \in \mathbb{N}$. This completes the proof. □

Recall that S_ω is the quotient space obtained by identifying all limit points of the topological sum of ω many convergent sequences.

Lemma 3.2. *([11]) A topological group G contains a closed copy of S_ω if and only if G contains a closed copy of S_2.*

Theorem 3.3. *Let X be a topological space. If $F(X)$ contains a non-trivial convergent sequence, then $F(X)$ contains a closed copy of S_2, equivalently, $F(X)$ contains a closed copy of S_ω.*

Proof. If $F(X)$ contains a non-trivial convergent sequence, then there is a non-trivial sequence $L = \{x_n\}_{n\in\mathbb{N}}$ converging to the identity e. By Lemma 2.6, $L \subset F_{n_0}(X)$ for some $n_0 \in \mathbb{N}$, which implies $l(x_n) \leq n_0$ for every $n \in \mathbb{N}$. By Lemma 3.1, there is a sequence $\{t_k\}_{k\in\mathbb{N}}$ converging to e and the length of every t_k is greater than $2n_0$. Thus $\{x_1 t_k\}_{k\in\mathbb{N}}$ converges to x_1, and $l(x_1 t_k) > n_0$ for every $k \in \mathbb{N}$. Put $y_{1,k} = x_1 t_k$ for every $k \in \mathbb{N}$ and $L_1 = \{y_{1,k}\}_{k\in\mathbb{N}}$. Using again Lemma 2.6, we can choose $n_1 \in \mathbb{N}$ such that the length of every element in L_1 is less than n_1. By induction, we can choose a sequence $\{n_i\}_{i\in\mathbb{N}}$ with $n_1 < n_2 < \cdots$, and a sequence $\{L_i\}_{i\in\mathbb{N}}$ with $L_i = \{y_{i,k}\}_{k\in\mathbb{N}}$ converging to x_i and $n_{i-1} < l(y_{i,k}) < n_i$ for every $i, k \in \mathbb{N}$. Put

$$S = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_{n,k} : n, k \in \mathbb{N}\}.$$

Claim. S is closed in $F(X)$ and is a copy of S_2.

Let $f \in \mathbb{N}^{\mathbb{N}}$. The set $\bigcup_{n\in\mathbb{N}}\{y_{n,k} : k < f(n)\}$ is closed and discrete in $F(X)$ by Lemma 2.5. Then the set

$$\{e\} \cup \bigcup_{n\geq i}\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$$

is an open neighborhood of e in S for every $i \in \mathbb{N}$. It is also easy to see that $\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$ is open in S for every $n \in \mathbb{N}$, and $\{y_{n,k}\}$ is open in S for every $n, k \in \mathbb{N}$. Hence the space S is a copy of S_2.

Now we will show that S is closed in $F(X)$. Suppose $p \notin S$. Since $\{e\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, there exist open subsets U and V of $F(X)$ such that

$$p \in U, \{e\} \cup \{x_n : n \in \mathbb{N}\} \subset V \text{ and } U \cap V = \emptyset.$$

Thus there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{e\} \cup \bigcup_{n\in\mathbb{N}}\{x_n\} \cup \{y_{n,k} : k \geq f(n)\} \subset V.$$

Let

$$W = U \setminus \bigcup_{n\in\mathbb{N}}\{y_{n,k} : k < f(n)\}.$$

The set W is an open neighborhood of p in $F(X)$ by Lemma 2.5 and $W \cap S = \emptyset$, whence S is closed in $F(X)$. □

Remark 3.4. E. Ordman, B. Smith-Thomas asked whether X contains a non-trivial convergent sequence if $F(X)$ contains a non-trivial convergent sequence [12, Question 3.11]. M. Tkachenko constructed a topological space X without infinite compact subsets such that $F(X)$ contains a non-trivial convergent sequence [14, Theorem 3.5]. Thus the answer to [12, Question 3.11] is negative.

Corollary 3.5. *Let X be a topological space. If $F(X)$ is a sequential space, then either X is discrete or $F(X)$ contains a closed copy of S_2.*

Proof. If X is not discrete, then $F(X)$ is also not discrete. $F(X)$ contains a non-trivial convergent sequence, since $F(X)$ is a sequential space. By Theorem 3.3, $F(X)$ contains a closed copy of S_2. □

Corollary 3.6. *Let X be a topological space. If $F(X)$ is a sequential space, then either X is discrete or $F(X)$ contains a closed copy of S_ω.*

Corollary 3.7. *([12]) Let X be a topological space. If $F(X)$ is a Fréchet–Urysohn space, then the space X is discrete.*

Remark 3.8. There exists a sequential space X that contains no a copy of S_2 or S_ω, but is not Fréchet [5, Example 2.14].

Now we further strengthen Corollary 3.7 by discussing κ-Fréchet property.

Let $\mathbb{E}, \mathbb{O}$ be positive even, odd number sets, respectively.

Theorem 3.9. *Let X be a non-discrete space, $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $A \subset \mathbb{E}$, then $\bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is dense in $\bigcup_{i\in\mathbb{N}} E_{2i}(X)$; if $A \subset \mathbb{O}$, then $\bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is dense in $\bigcup_{i\in\mathbb{N}} E_{2i-1}(X)$.*

Proof. The space X being non-discrete, let x be an accumulation point of X, thus the identity e is an accumulation point of $H = \{ax^{-1} : a \in X\}$ in $F(X)$. Fix $p \in \bigcup_{i\in\mathbb{N}} E_{2i}(X)$ and an open neighborhood U of p in $F(X)$, there is an open neighborhood V at e such that $pV \subset U$. Then $l(p) = 2i_0$ for some $i_0 \in \mathbb{N}$. Put

$$n_j = \min\{n_i \in A : n_i > l(p), i \in \mathbb{N}\}.$$

Hence, $n_j - l(p) = 2k$ for some $k \in \mathbb{N}$. Let W be an open neighborhood of e in $F(X)$ such that $W^k \subset V$. Then W contains infinitely many elements of H, which implies that there exists $K = \{a_n : n = 1, 2, ..., k\} \subset X$ such that

$$(\{x\} \cup \mathrm{supp}(p)) \cap K = \emptyset \text{ and } \{a_n x^{-1} : n \le k\} \subset W \cap H.$$

Let $y = pa_1x^{-1}a_2x^{-1} \cdots a_kx^{-1}$, then $y \in pW^k \subset pV \subset U$. We have $y \in E_{n_j}(X)$ since $l(y) = l(p) + 2k = n_j$. Hence $U \cap \bigcup_{i\in\mathbb{N}} E_{n_i}(X) \neq \emptyset$. The second part can be proved in a similar fashion. □

Corollary 3.10. *Let X be a non-discrete space, $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $|A \cap \mathbb{E}| = \omega$ and $|A \cap \mathbb{O}| = \omega$, then $\bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is dense in $F(X)$.*

Theorem 3.11. *Let X be a topological space and $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is κ-Fréchet–Urysohn and contains no copy of S_2, then X is discrete.*

Proof. Suppose that X is non-discrete. Without loss of generality, we assume that $A \subset \mathbb{E}$ and $n_1 < n_2 < \cdots$. Lemma 2.4 implies that $\bigcup_{i>k} E_{n_i}(X)$ is open in C for every $k \in \mathbb{N}$. Choose $x \in E_{n_1}(X)$. By Theorem 3.9, $x \in \overline{\bigcup_{i>1} E_{n_i}(X)}$. Since C is κ-Fréchet–Urysohn, there is a non-trivial sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $\bigcup_{i>1} E_{n_i}(X)$ converging to x. According to Lemma 2.6, $\{x_n : n \in \mathbb{N}\} \cup \{x\} \subset F_{i_0}(X)$ for some $i_0 \in \mathbb{N}$. Again, by Theorem 3.9, $x_1 \in \overline{\bigcup_{i>i_0} E_{n_i}(X)}$. There exist $i_1 > i_0$ and a non-trivial sequence $\{x_{1,k}\}_{k\in\mathbb{N}}$ of points of $\bigcup_{i_0<i<i_1} E_{n_i}(X)$ converging to x_1; similarly, there exist $i_2 > i_1$ and a non-trivial sequence $\{x_{2,k}\}_{k\in\mathbb{N}}$ of points of $\bigcup_{i_1<i<i_2} E_{n_i}(X)$ converging to x_2. In this way, we can choose a sequence $\{i_k\}_{k\in\mathbb{N}}$ with $i_k > i_{k-1}$ and a non-trivial sequence $\{x_{m,k}\}_{k\in\mathbb{N}}$ of points of $\bigcup_{i_{m-1}<i<i_m} E_{n_i}(X)$ converging to x_m. It follows from the proof of Theorem 3.3 that

$$\{x\} \cup \{x_m : m \in \mathbb{N}\} \cup \{x_{m,k} : m, k \in \mathbb{N}\}$$

is a copy of S_2. This contradicts the hypothesis that C contains no copy of S_2. □

Corollary 3.12. *([15]) Let $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is Fréchet–Urysohn, then X is discrete.*

4. The Fréchet property and S_2 of $F_5(X)$ and $F_4(X)$

Next, we shall considerably improve Theorem 1.3 in the introduction.

Theorem 4.1. *Let X be a μ-space. If $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete.*

Proof. Suppose X is neither discrete nor compact. Since X is a μ-space, X is not countably compact. Then in X there exist a non-trivial convergent sequence C containing its limit point and an infinite countable discrete closed subset $D \subset X$ such that $C \cap D = \emptyset$.

Claim. $F_5(X)$ contains a copy of S_2.

In fact, let $C = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and $D = \{d_n : n \in \mathbb{N}\}$, where $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$. Thus $\{x^{-1}x_k\}_{k\in\mathbb{N}}$ converges to $e \in F(X)$. Let $y_{n,k} = x_n d_n x^{-1} x_k d_n^{-1}$ for every $n, k \in \mathbb{N}$. Then $\{y_{n,k}\}_{k\in\mathbb{N}}$ converges to $x_n \in F(X)$ for every $n \in \mathbb{N}$.

Put

$$L = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_{n,k} : n, k \in \mathbb{N}\}.$$

Obviously, $L \subset F_5(X)$ and L is a sequential space. We shall show that L is a copy of S_2.

Fix two subsequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ with $n_1 < n_2 < \cdots$ and $k_1 \leq k_2 \leq \cdots$. Then the sequence $\{y_{n_i,k_{n_i}}\}_{i\in\mathbb{N}}$ does not converge in $F(X)$. Otherwise, by Lemma 2.8, $\{d_{k_{n_i}} : i \in \mathbb{N}\} \subset \overline{\mathrm{supp}(\{y_{n_i,k_{n_i}} : i \in \mathbb{N}\})}$ is compact in X. This contradicts the fact that $\{d_{k_{n_i}} : i \in \mathbb{N}\}$ is an infinite subset of D.

Let $f \in \mathbb{N}^{\mathbb{N}}$. Then the set $\bigcup_{n\in\mathbb{N}}\{y_{n,k} : k < f(n)\}$ is closed and discrete in $F_5(X)$ and the set

$$\{x\} \cup \bigcup_{n\geq i}\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$$

is an open neighborhood of x in L for every $i \in \mathbb{N}$. It is also easy to see that $\{x_n\} \cup \{y_{n,k} : k \geq f(n)\}$ is open in L for every $n \in \mathbb{N}$, and $\{y_{n,k}\}$ is open in L for every $n, k \in \mathbb{N}$. Hence the space L is a copy of S_2. This completes the proof of Claim.

By above Claim, S_2 is a Fréchet–Urysohn space. This is a contradiction. □

The following theorem was proved not long ago [16].

Theorem 4.2. *([16]) Let X be a locally compact, metrizable space and the set of all non-isolated points of X is compact. If $F_4(X)$ is a k-space, then $F_4(X)$ is a Fréchet–Urysohn space.*

In this paper, we present a shorter alternative proof for Theorem 4.2 using Tanaka's theorem (Theorem 2.2). We still need a few auxiliary lemmas.

Lemma 4.3. *Let X be a metrizable space and $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of reduced elements of $F(X)$ with the length $n \in \mathbb{N}$, where $x_k = x_{k,1} \cdots x_{k,n}$ for every $k \in \mathbb{N}$ and $x_{k,i} \in X \cup X^{-1}$ for every $i \leq n$. If $\{x_k\}_{k\in\mathbb{N}}$ converges to $a \in F(X)$, then there is a sequence $\{k_j\}_{j\in\mathbb{N}}$ with $k_1 < k_2 < \cdots$ such that $\{x_{k_j,i}\}_{j\in\mathbb{N}}$ converges to some $a_i \in X \cup X^{-1}$ for every $i \leq n$ and $a = a_1 \cdots a_n$.*

Proof. Since $\{x_k : k \in \mathbb{N}\} \cup \{a\}$ is bounded in $F(X)$, it follows from Lemma 2.8 that the closure $\overline{\mathrm{supp}(\{x_k : k \in \mathbb{N}\} \cup \{a\})}$ of $\mathrm{supp}(\{x_k : k \in \mathbb{N}\} \cup \{a\})$ in X is compact. Let

$$Z = \overline{\mathrm{supp}(\{x_k : k \in \mathbb{N}\} \cup \{a\})} \cup \overline{\mathrm{supp}(\{x_k : k \in \mathbb{N}\} \cup \{a\})}^{-1}.$$

Then Z is a compact subset of $\tilde{X} = X \oplus \{e\} \oplus X^{-1}$, so is Z^n in $\tilde{X}^n$. Thus $p_k = (x_{k,1}, ..., x_{k,n}) \in Z^n \cap i_n^{-1}(x_k)$ for every $k \in \mathbb{N}$. Then $\{p_k\}_{k\in\mathbb{N}}$ has a subsequence $\{p_{k_j}\}_{j\in\mathbb{N}}$ converging to some $(a_1, a_2, ..., a_n) \in Z^n$, i.e., $\{x_{k_j,i}\}_{j\in\mathbb{N}}$ converges to $a_i \in X \cup X^{-1}$ for every $i \leq n$. It immediately follows from the continuity of the mapping i_n that $a = a_1 \cdots a_n$. □

Lemma 4.4. *Let $X = K \oplus D$, where K is a metrizable space and D is a discrete space. Suppose that $L = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,k} : n, k \in \mathbb{N}\}$ is a copy of S_2 in $F_4(X)$, where $l(x_{n,k}) = 4$ for every $n, k \in \mathbb{N}$. Then there exists a copy L_1 of S_2 such that $L_1 \subset L$ and $|supp(L_1) \cap D| < \omega$.*

Proof. We write $x_{n,k} = a_{n,k}b_{n,k}c_{n,k}d_{n,k}$, where $a_{n,k}, b_{n,k}, c_{n,k}, d_{n,k} \in X \cup X^{-1}$ for every $n, k \in \mathbb{N}$. Since $\{x_{n,k}\}_{k\in\mathbb{N}}$ converges to x_n for every $n \in \mathbb{N}$, by Lemma 4.3, we obtain that $\{a_{n,k_j}\}_{j\in\mathbb{N}}$ converges to some

$a_n \in X \cup X^{-1}$, $\{b_{n,k_j}\}_{j\in\mathbb{N}}$ converges to some $b_n \in X \cup X^{-1}$, $\{c_{n,k_j}\}_{j\in\mathbb{N}}$ converges to some $c_n \in X \cup X^{-1}$, $\{d_{n,k_j}\}_{j\in\mathbb{N}}$ converges to some $d_n \in X \cup X^{-1}$, and so $x_n = a_n b_n c_n d_n$.

Without loss of generality, we may assume that $l(x_n) = 4$ for every $n \in \mathbb{N}$ or $l(x_n) = 2$ for every $n \in \mathbb{N}$.

Case 1. $l(x_n) = 4$ for every $n \in \mathbb{N}$.

Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $F(X)$, using again Lemma 4.3, we have $\{a_{n_i}\}_{i\in\mathbb{N}}$ converges to some $a \in X \cup X^{-1}$, $\{b_{n_i}\}_{i\in\mathbb{N}}$ converges to some $b \in X \cup X^{-1}$, $\{c_{n_i}\}_{i\in\mathbb{N}}$ converges to some $c \in X \cup X^{-1}$, $\{d_{n_i}\}_{i\in\mathbb{N}}$ converges to some $d \in X \cup X^{-1}$, and so $x = abcd$.

If $a \in K\cup K^{-1}$, since $K\cup K^{-1}$ is open in $X\cup X^{-1}$, then $\{a_{n_i} : i \geq i_0\} \subset K\cup K^{-1}$ for some $i_0 \in \mathbb{N}$. Without loss of generality, we may assume that $a_{n_i} \in K \cup K^{-1}$ for every $i \in \mathbb{N}$. Further, since $\{a_{n_i,k_j}\}_{j\in\mathbb{N}}$ converges to a_{n_i}, we have $\{a_{n_i,k_j} : j \geq j_0\} \subset K \cup K^{-1}$ for some $j_0 \in \mathbb{N}$. We may assume that $a_{n_i,k_j} \in K \cup K^{-1}$ for every $j \in \mathbb{N}$. If $a \in D\cup D^{-1}$, since $\{a\}$ is open in $X\cup X^{-1}$, then there exists an $m \in \mathbb{N}$ such that $a_{n_i} = a$ for every $i \geq m$. We may assume that $a_{n_i} = a$ for every $i \in \mathbb{N}$. Further, since $\{a_{n_i,k_j}\}_{j\in\mathbb{N}}$ converges to $a_{n_i} = a$, we may assume that $a_{n_i,k_j} = a$ for every $j \in \mathbb{N}$. The analogous statements are valid for b, c, and d.

Put

$$L_1 = \{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \cup \{x_{n_i,k_j} : i,j \in \mathbb{N}\}.$$

Then L_1 is a copy of S_2 such that $L_1 \subset L$ and $|\mathrm{supp}(L_1) \cap D| < \omega$.

Case 2. $l(x_n) = 2$ for every $n \in \mathbb{N}$.

This means that we may assume $a_n = b_n^{-1}$ for every $n \in \mathbb{N}$ or $b_n = c_n^{-1}$ for every $n \in \mathbb{N}$ or $c_n = d_n^{-1}$ for every $n \in \mathbb{N}$. We only prove the case that $a_n = b_n^{-1}$ for every $n \in \mathbb{N}$. The arguments for the rest are similar.

Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $F(X)$, i.e., $\{c_n d_n\}_{n\in\mathbb{N}}$ converges to x in $F(X)$, by Lemma 4.3, we have $\{c_{n_i}\}_{i\in\mathbb{N}}$ converges to some $c \in X \cup X^{-1}$, $\{d_{n_i}\}_{i\in\mathbb{N}}$ converges to some $d \in X \cup X^{-1}$, and so $x = cd$.

If $c \in K \cup K^{-1}$, then $\{c_{n_i} : i \geq i_0\} \subset K \cup K^{-1}$ for some $i_0 \in \mathbb{N}$. Without loss of generality, we may assume that $c_{n_i} \in K \cup K^{-1}$ for every $i \in \mathbb{N}$. Further, since $\{c_{n_i,k_j}\}_{j\in\mathbb{N}}$ converges to c_{n_i}, we have $\{c_{n_i,k_j} : j \geq j_0\} \subset K \cup K^{-1}$ for some $j_0 \in \mathbb{N}$. We may assume that $c_{n_i,k_j} \in K \cup K^{-1}$ for every $j \in \mathbb{N}$. If $c \in D \cup D^{-1}$, then $c_{n_i} = c$ for some $i \geq i_0$. We may assume that $c_{n_i} = c$ for every $i \in \mathbb{N}$. Further, since $\{c_{n_i,k_j}\}_{j\in\mathbb{N}}$ converges to $c_{n_i} = c$, we may assume that $c_{n_i,k_j} = c$ for every $j \in \mathbb{N}$. The analogous statements are valid for d.

If $a_n \in D \cup D^{-1}$, then $b_n \in D \cup D^{-1}$. Since $\{a_{n,k_j}\}_{j\in\mathbb{N}}$ converges to a_n and $\{b_{n,k_j}\}_{j\in\mathbb{N}}$ converges to b_n, we may choose some $j \in \mathbb{N}$ such that $a_{n,k_j} = a_n$ and $b_{n,k_j} = b_n$, whence $l(x_{n,k_j}) = l(a_{n,k_j} b_{n,k_j} c_{n,k_j} d_{n,k_j}) = l(c_{n,k_j} d_{n,k_j}) < 4$. This is a contradiction. So we have $a_n, b_n \in K \cup K^{-1}$ for every $n \in \mathbb{N}$. Especially, $a_{n_i}, b_{n_i} \in K \cup K^{-1}$ for every $i \in \mathbb{N}$ Further, we may assume that $a_{n_i,k_j}, b_{n_i,k_j} \in K \cup K^{-1}$ for every $j \in \mathbb{N}$.

Put

$$L_1 = \{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \cup \{x_{n_i,k_j} : i,j \in \mathbb{N}\}.$$

Then L_1 is a copy of S_2 such that $L_1 \subset L$ and $|\mathrm{supp}(L_1) \cap D| < \omega$. □

Lemma 4.5. *Let $X = K \oplus D$, where K is a compact metrizable space and D is a discrete space. Then $F_4(X)$ contains no copy of S_2.*

Proof. By Theorem 1.2, $F_3(X)$ is Fréchet–Urysohn, and so contains no copy of S_2. Suppose that $L = \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,k} : n,k \in \mathbb{N}\}$ is a copy of S_2 in $F_4(X)$. Since $F_3(X)$ is closed in $F(X)$ and contains no copy of S_2, without loss of generality, we may assume $\{x_{n,k} : n,k \in \mathbb{N}\} \subset F_4(X) \setminus F_3(X)$. Applying Lemma 4.4, we can obtain a copy L_1 of S_2 such that $L_1 \subset L$ and $|\mathrm{supp}(L_1) \cap D| < \omega$.

Now, let $D_1 = \mathrm{supp}(L_1) \cap D$. Then $|D_1| < \omega$ and $L_1 \subset F(K \cup D_1, X)$, where $F(K \cup D_1, X)$ is the subgroup of $F(X)$ generated by $K \cup D_1$. Since $K \cup D_1$ is compact metrizable, by Lemma 2.7, $F(K \cup D_1, X)$

is topologically isomorphic to $F(K \cup D_1)$ and $F_4(K \cup D_1)$ is metrizable. So L_1 is metrizable. This is a contradiction. □

Lemma 4.6. *([10, Lemma 2.12]) Let X be a locally compact metrizable space and the set of all non-isolated points of X is compact. Then X can be expressed as a topological sum of a discrete subspace and a compact metrizable subspace.*

The Alternative Proof of Theorem 4.2. By Lemma 4.6, X can be expressed as a topological sum of a discrete subspace and a compact metrizable subspace. Then $F_4(X)$ contains no copy of S_2 by Lemma 4.5. Since X is a metrizable space, each singleton of $F_4(X)$ is a G_δ-set [2, Theorem 7.6.7]. By Theorem 2.2, $F_4(X)$ is a Fréchet–Urysohn space. This completes the proof. □

It is well known that if X is a locally compact, separable, metrizable space, then $F(X)$ is a k-space [3].

Corollary 4.7. *([16]) If X is a locally compact, separable, metrizable space, then the set of all non-isolated points of X is compact if and only if $F_4(X)$ is a Fréchet–Urysohn space.*

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