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A Normal and Moore Paratopological Group

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Abstract: In this paper, we prove the following: under MA+(¬CH), there exists a non-metrizable, separable, normal and Moore paratopological group. Therefore the existence of a separable, normal, non-metrizable Moore paratopological group is independent of the usual axioms of Set Theorem.

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0 Introduction

A *paratopological group* G is a group G with a topology such that the multiplication in G is jointly continuous. If G is a paratopological group and the inverse operation of G is continuous, then G is called a *topological group*. There exists a paratopological group which is not a topological group; the Sorgenfrey line is such an example[4]. Paratopological groups were discussed and many results have been obtained[1−2, 10, 12−16].

Our work in this paper is provoked by the following celebrated theorems in the theory of topological groups.

Theorem 0.1[3, 8] (Birkhoff-Kakutani Theorem) Each first-countable topological group is metrizable.

In particular, each Moore topological group is metrizable.

Theorem 0.2[17] If a topological group G is a p-space, then it is paracompact.

However, in the class of paratopological groups, the situation changes dramatically. In [9] and [13], the authors independently gave the following example:

Example 0.1[9, 13] There exists a Tychonoff Moore paratopological group which is not normal, and hence it is not paracompact.

It is noteworthy that the above group is a non-normal Moore space[19]. The following question is natural: Is a normal Moore paratopological group metrizable? This makes us think

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of the famous "Normal Moore Space Conjecture". Under what circumstances does normality imply paracompactness? On the other hand, each Tychonoff Moore space is a p-space and a σ-space[6]. Hence, Lin[11] posed the following question:

Question 0.1[11] Let G be a normal paratopological group. If G is a σ-space and a k-space, is G paracompact?

In this paper, we construct the following example:

Example 0.2 (MA+($\neg$CH)) There exists a non-metrizable, separable, normal and Moore paratopological group.

We denote by $\mathbb{Q}$ and $\mathbb{R}$ the sets of all rational numbers and real numbers, respectively. The letters ω and $\mathfrak{c}$ denote the first infinite ordinal and the cardinality of the continuum, respectively. Readers may consult some books[2, 4, 6] for notations and terminology not explicitly given here.

1 Construction of Example 0.2

In order to construct Example 0.2, we first have to prove a crucial theorem.

For each $p=(p_1,p_2)\in\mathbb{R}^2$, $\varepsilon>0$, let $B(p,\varepsilon)$ be the open ball with center p and radius ε in $\mathbb{R}^2$, and put $B'(p,\varepsilon)=\{p\}\cup B((p_1,p_2+\varepsilon),\varepsilon)$. Clearly, the disc $B((p_1,p_2+\varepsilon),\varepsilon)$ is tangent to the line $y=p_2$ at the point p. Then $B'(p,\varepsilon)$ is called the tangent disc neighborhood of p or, for convenience, an open tangent disc with radius ε at point p.

We define a topology on $\mathbb{R}^2$ by taking a local base at each point $p\in\mathbb{R}^2$ as follows: $\{B'(p,\varepsilon):\varepsilon>0\}$. It is called a *generalized Niemytzki's tangent disc topology* on $\mathbb{R}^2$, and the $\mathbb{R}^2$ endowed with this topology is called a *generalized Niemytzki's tangent disc space*.

In this paper, we always denote by $\mathcal{D}$ and $\mathcal{E}$ the generalized Niemytzki's tangent disc topology and Euclidean topology on $\mathbb{R}^2$, respectively. Let M be a subset of $\mathbb{R}^2$. We always denote by $\mathcal{D}|_M$ and $\mathcal{E}|_M$ the subset M endowed with the subspace topology of the generalized Niemytzki's tangent disc topology and Euclidean topology on $\mathbb{R}^2$, respectively.

A space X is called a *Q-set* if each subset of it is an F_σ-set in X. The first and most famous example of a normal non-metrizable Moore space can be obtained from a Q-set[18]. The existence of an uncountable Q-set in $\mathbb{R}$ is actually equivalent to the existence of a separable normal non-metrizable Moore space[7].

Theorem 1.1 Let M be a subset of $\mathbb{R}^2$. If $(M,\mathcal{E}|_M)$ is a Q-set, then $(M,\mathcal{D}|_M)$ is a normal space.

Proof Clearly, $(M,\mathcal{D}|_M)$ is a T_2-space. Let A,B be two disjoint closed sets in $(M,\mathcal{D}|_M)$. Since $(M,\mathcal{E}|_M)$ is a Q-set, it is obvious that A,B are F_σ-sets in $(M,\mathcal{E}|_M)$, and hence there exist two sequences of closed sets $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ in $\mathcal{E}$ such that $A=\bigcup_{n\in\omega}(A_n\cap M)$ and $B=\bigcup_{n\in\omega}(B_n\cap M)$.

Take $n\in\omega$. If $x\in A_n\cap M$, then there exists ε_x such that $0<\varepsilon_x<1$ and $B'(x,3\varepsilon_x)\cap B=\emptyset$ since B is closed in $(M,\mathcal{D}|_M)$. Put $U_n=(\bigcup\{B'(x,\frac{\varepsilon_x}{2}):x\in A_n\cap M\})\cap M$. Then U_n is open in

$(M, \mathcal{D}|_M)$ and contains A_n. We first prove the following claim.

Claim We have $\overline{U_n} \cap B = \emptyset$ in $(M, \mathcal{D}|_M)$.

Fix an arbitrary point $y \in B$. Since $y \notin A_n$ and A_n is closed in $\mathcal{E}$, there exists δ, $0 < \delta < \frac{1}{2}$, such that $B(y, 2\delta) \cap A_n = \emptyset$. In order to complete the proof of the claim, we only need to prove the following subclaim.

Subclaim For each $x \in A_n \cap M$, we have $B'(y, \frac{\delta^2}{2}) \cap B'(x, \frac{\varepsilon_x}{2}) = \emptyset$.

Let d be the Euclidean metric in $\mathbb{R}^2$, and denote by $S(z, \varepsilon)$ the circumference with center z and radius ε in $\mathbb{R}^2$. Since $x \notin B(y, 2\delta)$, we have $d(x, y) \geq 2\delta$. Let the second coordinates of x and y be x_2 and y_2, respectively.

Case 1 $y_2 \leq x_2$.

If $x_2 \geq y_2 + \delta^2$, then $B(y, \delta^2) \cap B'(x, \frac{\varepsilon_x}{2}) = \emptyset$. Hence we may assume that $y_2 \leq x_2 < y_2 + \delta^2$. Since $\delta^2 < 2\delta$ and $x \notin B(y, 2\delta)$, we can take a point $u \in S(y, 2\delta)$ such that the line segment $\overline{xu}$ is parallel to x-axis and the points x, u are located on the same side of $S(y, 2\delta)$ (possibly, we have $u = x$). At the same time, we can choose another point v on the other side of $S(y, 2\delta)$ such that the line segment $\overline{yv}$ is parallel to x-axis (by the symmetry, we can also choose a point v such that x, u, v are located on the same side of $S(y, 2\delta)$. However, we did not do it for convenience's sake, see Fig. 1). Denote the centers of $B'(x, \frac{\varepsilon_x}{2})$, $B'(u, \frac{\varepsilon_x}{2})$ and $B'(v, \frac{\varepsilon_x}{2})$ by x', u' and v', respectively. Denote the center of the tangent disc $B'(v, \frac{1}{2})$ by v''.

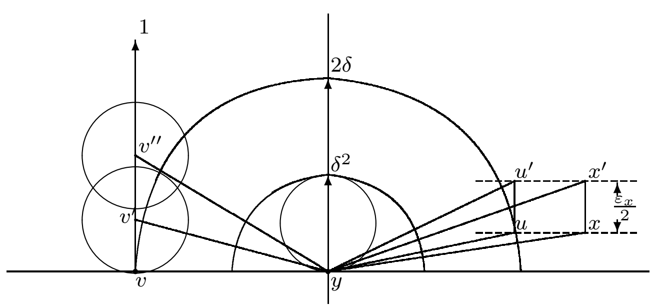

Figure 1 $y_2 \leq x_2$

Since $\delta < \frac{1}{2}$, it follows from the triangle $\triangle yvv''$ that

$$d(y, v'') - \frac{1}{2} = \sqrt{(2\delta)^2 + \left(\frac{1}{2}\right)^2} - \frac{1}{2} = \frac{4\delta^2}{\sqrt{(2\delta)^2 + (\frac{1}{2})^2} + \frac{1}{2}} > \frac{4\delta^2}{2\delta + 1} > 2\delta^2,$$

then $B(y, \delta^2) \cap B'(v, \frac{1}{2}) = \emptyset$. Moreover, since $\varepsilon_x < 1$, we have $B'(v, \frac{\varepsilon_x}{2}) \subset B'(v, \frac{1}{2})$. Hence, $B(y, \delta^2) \cap B'(v, \frac{\varepsilon_x}{2}) = \emptyset$. Then we claim that $d(y, x') \geq d(y, v') > \delta^2 + \frac{\varepsilon_x}{2}$. Indeed, it is obvious that $d(y, x') \geq d(y, u')$ and $d(y, v') > \delta^2 + \frac{\varepsilon_x}{2}$, so it suffices to show that $d(y, u') \geq d(y, v')$. It follows from the triangles $\triangle yvv'$ and $\triangle yuu'$ that

$$\begin{aligned} d(y,v') &= \sqrt{d(y,v)^2 + d(v',v)^2} \\ &\le \sqrt{d(y,v)^2 + d(v',v)^2 - 2d(y,v)d(v',v)\cos\angle yuu'} \\ &= d(y,u'). \end{aligned}$$

Therefore, $B(y,\delta^2) \cap B'(x,\frac{\varepsilon_x}{2}) = \emptyset$, and $B'(y,\frac{\delta^2}{2}) \cap B'(x,\frac{\varepsilon_x}{2}) = \emptyset$.

Case 2 $x_2 < y_2$.

If $x_2 + \varepsilon_x \le y_2$, then we have $B'(y,\frac{\delta^2}{2}) \cap B'(x,\frac{\varepsilon_x}{2}) = \emptyset$. Hence, we may assume that $x_2 < y_2 < x_2 + \varepsilon_x$. Put $\gamma_x = y_2 - x_2$. Then $0 < \gamma_x < \varepsilon_x$. Denote the centers of $B'(x,\frac{\varepsilon_x}{2})$ and $B'(x,3\varepsilon_x)$ by x' and x'', respectively. Let v be the projection of the point y to the line $\overline{xx''}$. Moreover, we can also assume without loss of generality that $B(y,2\delta) \cap B'(x,\frac{\varepsilon_x}{2}) \ne \emptyset$, thus $d(y,x') < 2\delta + \frac{\varepsilon_x}{2}$. It follows from items (a)−(c) below that $d(y,x') > \delta^2 + \frac{\varepsilon_x}{2}$.

(a) $d(y,x') \ge d(y,v)$, since the segment $\overline{yx'}$ is the hypothenuse in the triangle $\triangle yvx'$, while the segment $\overline{yv}$ is a leg; see Fig. 2.

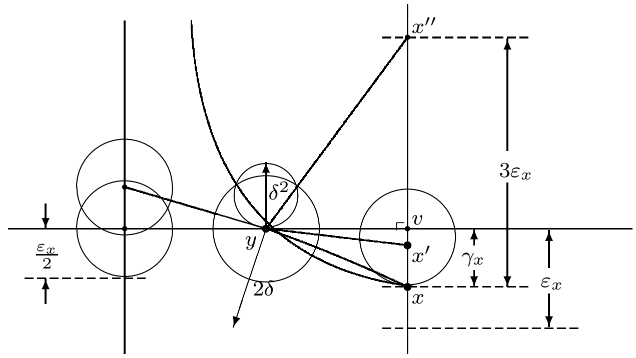

Figure 2 $x_2 < y_2$

(b) We can estimate the lower bound of $d(y,x')$ when $\gamma_x = \frac{\varepsilon_x}{2}$ (that is, $x' = v$). A calculation shows that $d(y,x') > \sqrt{3}\delta$. Indeed, we have $d(x,v) = \frac{\varepsilon_x}{2}$ and $d(v,x'') = 3\varepsilon_x - \frac{\varepsilon_x}{2}$. Since $y \notin B'(x,3\varepsilon_x)$, we have $d(y,x'') \ge 3\varepsilon_x$. In the triangle $\triangle yvx''$, we have

$$d(y,x'')^2 = d(y,v)^2 + d(v,x'')^2.$$

Since $d(y,v) = d(y,x') < 2\delta + \frac{\varepsilon_x}{2}$, it follows from the above formula that

$$(3\varepsilon_x)^2 < \left(2\delta + \frac{\varepsilon_x}{2}\right)^2 + \left(3\varepsilon_x - \frac{\varepsilon_x}{2}\right)^2,$$

that is, $\frac{11}{4}\varepsilon_x^2 < (2\delta + \frac{\varepsilon_x}{2})^2$, and hence $\varepsilon_x < 2\delta$. Therefore, we have

$$d(y,x') = d(y,v) = \sqrt{d(y,x)^2 - d(x,v)^2} \ge \sqrt{(2\delta)^2 - \left(\frac{\varepsilon_x}{2}\right)^2} > \sqrt{3}\delta$$

in the triangle $\triangle yvx$.

(c) Since $\gamma_x \in (0, \varepsilon_x)$, the second coordinate of x' is contained in $(y_2 - \frac{\varepsilon_x}{2}, y_2 + \frac{\varepsilon_x}{2})$. Then $d(y, x') - \frac{\varepsilon_x}{2}$ obtains the minimum value when $x' = v$. By the proof of (b), we see that $\varepsilon_x < 2\delta$. Since $\delta < \frac{1}{2}$, it follows from (b) that

$$d(y, x') - \frac{\varepsilon_x}{2} > \sqrt{3}\delta - \delta > \frac{\delta}{2} > \delta^2,$$

that is, $d(y, x') > \delta^2 + \frac{\varepsilon_x}{2}$.

Since $d(y, x') > \delta^2 + \frac{\varepsilon_x}{2}$, we have $B(y, \delta^2) \cap B'(x, \frac{\varepsilon_x}{2}) = \emptyset$, thus we have $B'(y, \frac{\delta^2}{2}) \cap B'(x, \frac{\varepsilon_x}{2}) = \emptyset$ by $B'(y, \frac{\delta^2}{2}) \subset B(y, \delta^2)$.

By Cases 1 and 2, Subclaim holds.

Therefore, it follows from Claim that $\overline{U_n} \cap B = \emptyset$ in $(M, \mathcal{D}|_M)$. Similarly, there exists an open set $V_n \supset B_n$ in $(M, \mathcal{D}|_M)$ such that $\overline{V_n} \cap A = \emptyset$.

Put

$$U = \bigcup_{n \in \omega} \Big(U_n \setminus \bigcup_{j \le n} \overline{V_j} \Big), \quad V = \bigcup_{n \in \omega} \Big(V_n \setminus \bigcup_{j \le n} \overline{U_j} \Big).$$

Then U and V are disjoint open sets in $(M, \mathcal{D}|_M)$ such that $A \subset U$ and $B \subset V$. Therefore, $(M, \mathcal{D}|_M)$ is normal. □

Lemma 1.1[5] (MA) Let X be a T_1-space having a σ-point-finite base. If $Y \subset X$ with cardinality $|Y| < \mathfrak{c}$, then Y is a Q-set.

Finally, we can construct Example 0.2.

Construction of Example 0.2 Let κ be a cardinal such that $\omega < \kappa < \mathfrak{c}$. Take a subset X of $\mathbb{R}$ such that $|X| = \kappa$ and $\mathbb{Q} \subset X$. Without loss of generality, we may assume that X is an additive subgroup of $\mathbb{R}$. Let $G = X \times \mathbb{Q}$. Then $(G, \mathcal{D}|_G, +)$ is a separable and Moore paratopological group, see [9] or [13]. Clearly, $(G, \mathcal{D}|_G)$ is not paracompact. By Theorem 1.1 and Lemma 1.1, $(G, \mathcal{D}|_G)$ is normal.

Example 0.2 gives a negative answer to Question 0.1 under MA+(¬CH). It is known that under CH every separable normal Moore space is metrizable. Thus under CH every separable normal Moore paratopological gruop is metrizable. Therefore the existence of a separable normal non-metrizable Moore paratopological group is independent of the usual axioms of Set Theorem. However, the answers to the following questions are still unknown.

Question 1.1 Under MA+(¬CH), is each metacompact normal paratopological group metrizable?

Question 1.2 Under CH, if G is a normal paratopological group that is a σ-space and a k-space, is G paracompact?

Question 1.3 Under CH, is each normal Moore paratopological group metrizable?

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References

[1] Arhangel'skii, A.V. and Reznichenko, E.A., Paratopological and semitopological groups versus topological groups, *Topology Appl.*, 2005, 151(1/2/3): 107-119.

[2] Arhangel'skii, A.V. and Tkachenko, M., Topological Groups and Related Structures, Paris: Atlantis Press, 2008.

[3] Birkhoff, G., A note on topological groups, *Compos. Math.*, 1936, 3: 427-430.

[4] Engelking, R., General Topology (Revised and Completed Edition), Berlin: Heldermann Verlag, 1989.

[5] Fleissner, W.G. and Reed, G.M., Paralindelöf spaces and spaces with a σ-locally countable base, *Topology Proc.*, 1977, 2(1): 89-110.

[6] Gruenhage, G., Generalized metric spaces, In: Handbook of Set-Theoretic Topology (Kunen, K. and Vaughan, J.E. eds.), Amsterdam: North-Holland, 1984, 423-501.

[7] Heath, R.W., Screenability, pointwise paracompactness and metrization of Moore spaces, *Canad. J. Math.*, 1964, 16: 763-770.

[8] Kakutani, S., Über die Metrisation der topologischen Gruppen, *Proc. Imperial Acad. Tokyo*, 1936, 12(4): 82-84 (in German).

[9] Li, P.Y., Mou, L. and Wang, S.Z., Notes on questions about spaces with algebraic structures, *Topology Appl.*, 2012, 159(17): 3619-3623.

[10] Lin, F.C., A note on free paratopological groups, *Topology Appl.*, 2012, 159(17): 3596-3604.

[11] Lin, F.C., A note on paratopological groups with countable networks of sequential neighborhoods, *Topology Proc.*, 2013, 41: 9-16.

[12] Lin, F.C. and Lin, S., Pseudobounded or ω-pseudobounded paratopological groups, *Filomat*, 2011, 25(3): 93-103.

[13] Lin, F.C. and Liu, C., On paratopological groups, *Topology Appl.*, 2012, 159(10/11): 2764-2773.

[14] Liu, C., A note on paratopological groups, *Comment. Math. Univ. Carolin.*, 2006, 47(4): 633-640.

[15] Liu, C., Metrizability of paratopological (semitopological) groups, *Topology Appl.*, 2012, 159(5): 1415-1420.

[16] Liu, C. and Lin, S., Generalized metric spaces with algebraic structures, *Topology Appl.*, 2010, 157(12): 1966-1974.

[17] Roelcke, W. and Dierolf, S., Uniform Structures on Topological Groups and Their Quotients, New York: McGraw-Hill, 1981.

[18] Tall, F.D., Normality versus collectionwise normality, In: Handbook of Set-Theoretic Topology (Kunen, K. and Vaughan, J.E. eds.), Amsterdam: North-Holland, 1984, 685-732.

[19] Tkachenko, M., Paratopological and semitopological groups versus topological groups, In: Recent Progress in General Topology III (Hart, K.P., Van Mill, J. and Simon, P. eds.), Paris: Atlantis Press, 2014, 825-882.

一个正规的 Moore 仿拓扑群

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摘要: 本文证明了在假设 MA+(¬CH) 下, 存在一个不可度量化的、可分的、正规的 Moore 仿拓扑群. 因此, 存在一个可分的、正规的、不可度量化的 Moore 仿拓扑群独立于一般集论公理.

关键词: 仿拓扑群; Moore 空间; 正规空间; Q 集