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A STUDY OF PSEUDOBASES

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ABSTRACT. In this paper, we discuss the relationship between pseudobases and k-networks. It is shown that a regular space with a point countable pseudobase is an $\aleph_0$-space, and that a regular space with a σ-hereditarily closure-preserving (closed) pseudobase if and only if it either is an $\aleph_0$-space or is a σ-closed discrete space which all compact subsets are finite.

All spaces are T_1

Let X be a topological space. A collection $\mathcal{P}$ of subsets of X is a pseusobase for X[1] if whenever K is a compact subset of an open set U of X, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. A collection $\mathcal{P}$ of subsets of X is a k-network for X if whenever K is a compact subset of an open set U of X, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite subcollection $\mathcal{P}'$ of $\mathcal{P}$. In this paper, we discuss the relationship between pseudobases and k-networks. Obviously, a regular space with a countable(σ-closure-preserving, σ-cushioned pair) pseudobase is equivalent to the space with a countable(σ-closure-preserving, σ-cushioned pair) k-network. The following question raised: Is a regular space with a point countable(σ-hereditarily closure-preserving) pseudobase equivalent to the space with a point countable(σ-hereditarily closure-preserving) k-network ?

We answer this question nagetively by proving that a regular space with a point countable pseudobase has a countable pseudobase, and that a regular space with a σ-hereditarily closure-preserving pseudobase has a σ-locally finite k-network.

Theorem 1. A Hausdorff space with a point countable pseudobase has a countable pseudobase.

Proof. Suppose X is a Hausdorff space with a point countable pseudobase. Let $\mathcal{P}$ be a point countable pseudobase for X. For each $x \in X$, take $y \in X-\{x\}$. Put

$$\mathcal{G} = \{X-cl(P) : y \in P \subset cl(P) \subset X-\{x\},\ P \in \mathcal{P}\}.$$

Then it is a countable family of open subsets of X, and $x \in \bigcap \mathcal{G}$. If $z \in X-\{x\}$, there exists an open set V of X such that $\{z,y\} \subset V \subset cl(V) \subset X-\{x\}$ because X is Hausdroff. Then $\{z,y\} \subset P \subset V$ for some $P \in \mathcal{P}$. So $cl(P) \subset X-\{x\}$ and $z,y \in P$. Thus $z \notin \bigcap \mathcal{G}$, and $\{x\} = \bigcap \mathcal{G}$. Therefore each point of X is a G_δ in X. We assert that X is separable. In fact, take $x \in X$. There exists a countable family $\{V_n : n \in N\}$ of open sets of X such that $\{x\} = \bigcap \{V_n : n \in N\}$. Let $\{P \in \mathcal{P} : x \in P\} = \{P_m : m \in N\}$. For each $n,m \in N$, if $P_m - V_n = \emptyset$, let $a_{n,m} = x$; if $P_m - V_n \neq \emptyset$, take $a_{n,m} \in P_m - V_n$. Put

$$A = \{a_{n,m} : n,m \in N\} \cup \{x\}.$$

Then $cl(A) = X$. For each open subset G of X, we can assume that $G-\{x\} \neq \emptyset$. Take $y \in G-\{x\}$. There exists $n \in N$ such that $y \in X-V_n$. Since $\{y,x\} \subset V_n \cup G$, $\{y,x\} \subset P_m \subset V_n \cup G$ for some $m \in N$. So $y \in P_m - V_n \subset G$, and $a_{n,m} \in G$. Hence X is separable. Now, let $A = \{x_i : i \in N\}$ be a countable dense subset of X. Put $\mathcal{P}' = \{P \in \mathcal{P} : P \cap A \neq \emptyset\}$. The $\mathcal{P}'$ is countable. For a non-empty compact subset K of an open set

U of X, there exists $i \in N$ such that $x_i \in U$. So $K \cup \{x_i\} \subset U$. Thus there exists $P \in \mathcal{P}$ such that $K \cup \{x_i\} \subset P \subset U$; i.e. $P \in \mathcal{P}'$ and $K \subset P \subset U$. Hence $\mathcal{P}'$ is a countable pseudobase for X. This completes the proof of the theorem.

A regular space with a countable pseudobase is an $\aleph_0$-space. By Theorem 1, a regular space is an $\aleph_0$-space if and only if it has a point countable pseudobase. Since there exists a regular, non-separable space with a σ-locally finite k-network(for example, a non-separable metric space), there exists a regular space with σ-locally finite k-network which has not a point countable pseudobase. A collection $\mathcal{P}$ of subsets of X is a p-pseudobase for X if whenever K is a compact subset of $X-\{x\}$, there exists $P \in \mathcal{P}$ such that $K \subset P \subset X-\{x\}$. Is a regular space with a countable p-pseudobase an $\aleph_0$-space ? Let X be a regular, countable space which is not an $\aleph_0$-space [1, Example 12.4]. Let $X = \{x_n : n \in N\}$. For each $n \in N$, put $P_n = X-\{x_n\}$. Then $\{P_n : n \in N\}$ is a countable p-pseudobase for X, but X is not an $\aleph_0$-space.

In the second part of this paper, we discuss the spaces with a σ-hereditarily closure-preserving pseudobase. A collection $\mathcal{P}$ of subsets of a space X is hereditarily closure-preserving (HCP) if, whenever a subset $C(P) \subset P$ is chosen for each $P \in \mathcal{P}$, the resulting collection $\mathcal{C} = \{C(P); P \in \mathcal{P}\}$ is closure-preserving. A collection $\mathcal{P}$ of subsets of X is weakly hereditarily closure-preserving (WHCP) if, whenever a point $x(P) \in P$ is chosen for each $P \in \mathcal{P}$, the resulting set $\{x(P) : P \in \mathcal{P}\}$ is a closed discrete subspace of X. A space X is $\aleph_1$-compact if every closed discrete subspace of X is countable.

Lemma 1. An $\aleph_1$-compact space with a σ-WHCP k-network has a countable k-network.

Proof. Suppose X is an $\aleph_1$-compact space with a σ-WHCP k-network $\mathcal{P}$. Let $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$, where $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\mathcal{P}_n$ is WHCP. For each $n \in N$, put

$A_n = \{x \in X : \mathcal{P}_n$ is not point countable at $x\}$. Then $\{P - A_n : P \in \mathcal{P}_n\}$ is countable, and A_n is a countable closed discrete subspace of X. In fact, if $\{P - A_n : P \in \mathcal{P}_n\}$ is not countable, there exists $\{P_a : a < \omega_1\}$ such that the $(P_a - A_n)$'s are distinct and non-empty. For each $a < \omega_1$, take a point $x_a \in P_a - A_n$. Since $\mathcal{P}_n$ is WHCP and X is $\aleph_1$-compact, $\{x_a : a < \omega_1\}$ is countable. So there exists an uncountable subset A of ω_1 and $x \notin A_n$ such that $x_a = x$ for each $a \in A$, a contradiction. Hence $\{P - A_n : P \in \mathcal{P}_n\}$ is countable. If $Z = \{z_h \in A_n : h \in H\}$ with $|H| \leq \aleph_1$, Since $\mathcal{P}_n$ is not point countable at point z_h, by well-ordering principle and transfinite induction we can obtain a subcollection $\{P_h : h \in H\}$ of $\mathcal{P}_n$ such that $z_h \in P_h$ and the P_h's are distinct. Since $\mathcal{P}_n$ is WHCP, Z is a closed discrete subspace of X. By the $\aleph_1$-compactness of X, A_n is a countable closed discrete subspace of X. Therefore $\mathcal{P}_n' = \{P - A_n : P \in \mathcal{P}_n\} \cup \{\{x\} : x \in A_n\}$ is a countable collection. If K is a compact subset of an open set U of X, there exists $n \in N$ and a finite subcollection $\mathcal{P}_n{}^*$ of $\mathcal{P}_n$ such that $K \subset \bigcup \mathcal{P}_n{}^* \subset U$. Since every closed discrete subspace of a compact space is finite, $K \cap A_n$ is finite. So

$$\mathcal{P}_n{}^{**} = \{P - A_n : P \in \mathcal{P}_n{}^*\} \cup \{\{x\} : x \in K \cap A_n\}$$

is a finite subcollection of $\mathcal{P}_n{}'$, and $K \subset \mathcal{P}_n^{**} \subset U$. Hence $\bigcup_{n \in N} \mathcal{P}_n{}'$ is a countable k-network for X. This completes the proof of the Lemma.

Lemma 2. A space with a σ-WHCP pseudobase either has a countable pseudobase or is the space which all compact closed subsets are finite.

Proof. Suppose a space X has a σ-WHCH pseudobase. Let $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$ be a σ-WHCP pseudobase for X, where each $\mathcal{P}_n$ is WHCP. If X has not a countable pseudobase, and X has an infinite compact closed subset K, then X is not an $\aleph_1$-compact space by Lemma 1, and there exists a non-closed countable subset $C = \{x_n : n \in N\} \subset K$. Since X is not $\aleph_1$-compact, there exists a closed discrete subspace $A \subset X$ such that $|A| = \aleph_1$. Then

$|A-K| = \aleph_1$ because $K \cap A$ is finite. Let $A-K = \{x_a : a<\omega_1\}$. For each $a<\omega_1$, let $V_a = X-\{x_b : a \neq b<\omega_1\}$, then V_a is an open set of X and $K \cup \{x_a\} \subset V_a$. So there exists $n(a) \in N$ and $P_a \in \mathcal{P}_{n(a)}$ such that $K \cup \{x_a\} \subset P_a \subset V_a$. Thus there exists an uncountable subset Λ of ω_1 and $m \in N$ such that $n(a) = m$ when $a \in \Lambda$. Since $x_a \in P_a \subset X-\{x_b\}$ when $a \neq b$, the P_a's are distinct. So $\{P_a : a \in \Lambda\}$ is WHCP. Now, by $\{x_n : n \in N\} \subset \bigcap\{P_a : a \in \Lambda\}$, there exists mutually distinct $a_n \in \Lambda$ such that $x_n \in P_{a_n}$ for each $n \in N$. Hence $C = \{x_n : n \in N\}$ is closed, a contradiction. Therefore X either has a countable pseudobase or is a -space which all compact closed subsets are finite. This completes the proof of the Lemma.

A space X has $\aleph_1$-tightness if for each $A \subset X$ with $x \in cl(A)$, there exists $H \subset A$ with $|H| \leq \aleph_1$ and $x \in cl(H)$.

Corollary. If a space with a σ-WHCP pseudobase has $\aleph_1$-tightness, then it either has a countable pseudobase or is a σ-closed discrete space which all compact closed subsets are finite.

Proof. Suppose a space X with a σ-WHCP pseudobase has $\aleph_1$-tightness. Let $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$ be a σ-WHCP pseudobase for X, where each $\mathcal{P}_n$ is WHCP. We can assume that X has not countable pseudobase. Then for each $x \in X$, $\mathcal{P}$ is not point countable at x. In fact, suppose $\mathcal{P}$ is point countable at x for some $x \in X$. Since X has not a countable pseudobase, X is not an $\aleph_1$-compact space by Lemma 1. There exists a closed discrete subspace $\{x_a : a < \omega_1\} \subset X - \{x\}$. For each $a < \omega_1$, put $V_a = X - \{x_b : a \neq b < \omega_1\}$. Then V_a is an open subset of X and $\{x, x_a\} \subset V_a$. So $\{x, x_a\} \subset P_a \subset V_a$ for some $P_a \in \mathcal{P}$. Hence the P_a's are distinct, and $x \in P_a$ for each $a < \omega_1$, a contradiction. Therefore $\mathcal{P}$ is not point countable at each $x \in X$.

For each $n \in N$, put

$X_n = \{x \in X : \mathcal{P}_n$ is not point countable at $x\}$.

Then $X = \bigcup_{n \in N} X_n$. We will show that each X_n is a closed discrete subspace of X; i.e. if $A \subset X_n$, then A is closed in X. Since X has $\aleph_1$-tightness, it is sufficient to show that each subset A of X_n with $|A| \leq \aleph_1$ is closed discrete in X. We can assume that $A = \{x_a : a < \omega_1\}$. By transfinite induction, we can obtain a subcollection $\{P_a : a < \omega_1\}$ of $\mathcal{P}_n$ such that $x_a \in P_a$ and the P_a's are distinct. Since $\mathcal{P}_n$ is WHCP, A is closed discrete in X. Hence each X_n is a closed discrete subspace of X. According to Lemma 2, all compact subsets of X are finite. This completes the proof of the Corollary.

A collection $\mathcal{P}$ of subsets of a space X is a (modk)-network for X if there exists a covering $\mathcal{K}$ of X by compact sets such that, whenever $K \subset U$ with $K \in \mathcal{K}$ and U open in X, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. A space with a σ-locally finite closed (mod k)-network is called a strong Σ-space. A space (X, τ) is a β-space if there is a function $g: N \times X \rightarrow \tau$

such that (1) $x \in g(n,x)$; (2) if $x \in g(n,x_n)$, then the set $\{x_n : n \in N\}$ has a cluster point in X. Obviously, σ-spaces are strong Σ-spaces, and strong Σ-spaces are β-spaces.

Lemma 3. Suppose X is a space which all compact subsets are finite. Then X is a σ-closed discrete space if it satisfies any one of the following:

(a) X is a strong Σ-space.

(b) X is a regular β-space which each point is a G_δ in X.

Proof. (a) Suppose a space X is a strong Σ-space which all compact subsets are finite. Let $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$ be a σ-locally finite closed (modk)-network for X with respect to a covering $\mathcal{K}$ of X by non-empty compact sets, where each $\mathcal{P}_n$ is a locally finite family of closed subsets of X. We can assume that $\mathcal{P}$ is closed under finite intersections. For a $K \in \mathcal{K}$, let $\mathcal{P}(K)=\{P \in \mathcal{P} : K \subset P\}$. Then $\mathcal{P}(K)$ is countable. So $\mathcal{P}(K)=\{P_i : i \in N\}$, and put $K_n = \bigcap_{i \le n} P_i$. Then $K \subset K_n \in \mathcal{P}(K)$. We assert that there exists $n \in N$ such that K_n is finite. In fact, if each K_n is infinite, then there exists a subset $A = \{x_n : n \in N\} \subset X-K$ such that $x_1 \in K_1 - K$ and $x_{n+1} \in K_{n+1} - K \cup \{x_1, x_2, \ldots, x_n\}$ because K is finite. If $cl(A) \cap K = \emptyset$, there exists $P_i \in \mathcal{P}(K)$ such that $K \subset P_i \subset X - cl(A)$. So $x_i \in K_i \subset P_i \subset X - \{x_i\}$, a contradiction. Thus $(cl(A)-A) \cap K = cl(A) \cap K \neq \emptyset$. Take $x \in (cl(A)-A) \cap K$. Since K is finite, there exist open sets V,U of X such that $x \in V, K-\{x\} \subset U$ and $V \cap U = \emptyset$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i} : i \in N\} \subset V$ because x is a cluster point of $\{x_n\}$. For an open neighbourhood G of x in X, since $K \subset (V \cap G) \cup U$, there exists $P_m \in \mathcal{P}(K)$ such that $K \subset K_m \subset P_m \subset (V \cap G) \cup U$. So $x_{n_i} \in K_{n_i} \cap V \subset K_m \cap V \subset G$ when $i \geq m$. This proves that the sequence $\{x_{n_i}\}$ converges to x. Hence $\{x\} \cup \{x_{n_i} : i \in N\}$ is an infinite compact subset of X, a contradiction. Therefore there exists

$n \in N$ such that K_n is finite.

For each $n \in N$, put

$$\mathcal{F}_n = \{P \in \mathcal{P}_n : P \text{ is finite}\}$$
$$= \{\{x_{a,1}, x_{a,2}, \dots, x_{a,n(a)}\} : a \in A_n\}.$$

Then $\bigcup_{n\in N} \mathcal{F}_n$ is a σ-locally finite (modk)-network for X. So $X = \bigcup_{n\in N} (\bigcup \mathcal{F}_n)$. For each $n,m \in N$, put $M_{n,m} = \{x_{a,m} : a \in A_n\}$. Then $M_{n,m}$ is a closed discrete subspace of X, and $X = \bigcup_{n,m\in N} M_{n,m}$.

Hence X is a σ-closed discrete space.

(b) Suppose a regular space (X,τ) is a β-space which all compact subsets are finite, and which each point is a G_δ in X. Then there exists a function $g: N \times X \to \tau$ such that

(1) $x \in g(n+1,x) \subset cl(g(n+1,x)) \subset g(n,x)$;

(2) $\bigcap_{n\in N} g(n,x) = \{x\}$;

(3) if $x \in g(n,x_n)$, the sequence $\{x_n\}$ has a cluster point in X.

For each $n \in N$, put $X_n = \{x \in X : x \in \bigcap_{z\neq x} (X-g(n,z))\}$

For each $y \in X$, if $z \in g(n,y) \cap X_n$, then $z = y$.

Thus

$$g(n,y)\cap X_n = \begin{cases} \{y\}, & \text{if } y\in X_n \\ \emptyset, & \text{if } y\notin X_n . \end{cases}$$

So X_n is a closed discrete subspace of X. We will show that $X = \bigcup_{n\in N} X_n$. If $X-\bigcup_{n\in N} X_n \neq \emptyset$, take $x \in X-\bigcup_{n\in N} X_n$. For each $n\in N$, there exists $x_n\neq x$, such that $x\in g(n,x_n)$ because $x\notin X_n$. If $\{x_n : n\in N\}$ is finite, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}=x_{n_1}$ for each $i\in N$. Since $x\in g(n_i,x_{n_i})\subset g(i,x_{n_i}) = g(i,x_{n_1})$, $x\in \bigcap_{i\in N} g(i,x_{n_1})$, i.e. $x = x_{n_1}$, a contradiction. Thus $\{x_n : n\in N\}$ is infinite. We can assume that the x_n's are distinct. Let q be a cluster point of $\{x_n\}$ in X. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\in g(i,q)$ for each $i\in N$. Put $F_n = cl\{x_{n_i} : i\geq n\}$, then $F_{n+1}\subset g(n,q)$. Since

$x \in g(i, x_{n_i})$, the sequence $\{x_{n_i}\}$ has a cluster point in X. So $\emptyset \neq \bigcap_{n \in N} F_n \subset \{q\}$; i.e. q is a unique cluster point of $\{x_{n_i}\}$ in X. And since every subsequence of $\{x_{n_i}\}$ has a cluster point in X, the sequence $\{x_{n_i}\}$ converges to q. Hence $\{x_{n_i} : i \in N\} \cup \{q\}$ is an infinite compact subset of X, a contradiction. Therefore $X = \bigcup_{n \in N} X_n$, and X is a σ-closed discrete space. This completes the proof of the Lemma.

Theorem 2. The following are equivalent for a regular space X:

(a) X is a space with a σ-HCP closed pseudobase.

(b) X is a space with a σ-HCP pseudobase.

(c) X is an $\aleph_0$-space, or a σ-closed discrete space which all compact subspaces are finite.

Proof. (a)$\Rightarrow$(b) is obvious. If X is a regular space with a σ-HCP pseudobase, then X has a σ-closure-preserving net. So X is a σ-space. By

Lemma 2 and Lemma 3, (c) holds. If X is an $\aleph_0$-space, then X has a σ-HCP closed pseudobase. If X is a σ-closed discrete space which all compact subsets are finite, let $X = \bigcup_{n \in N} X_n$, where each X_n is a closed discrete subspace of X. For each $n \in N$, put $\mathcal{P}_n = \{P \subset \bigcup_{i \leq n} X_i : |P| \leq n\}$.

Then $\mathcal{P}_n$ is a HCP family of closed subsets of X. Since all compact subsets of X are finite, $\bigcup_{n \in N} \mathcal{P}_n$ is a σ-HCP closed pseudobase for X. (a) holds.

A regular space with a σ-locally finite k-network is called an $\aleph$-space.

Corollary. A regulary space with a σ-HCP pseudobase is an $\aleph$-space.

Let X be a non-separable metric space. Then $Z = [0,1] \oplus X$ is an $\aleph$-space, but it hasnot a σ-HCP pseudobase by Lemma 2.

Example. There exists a regular space X with a closure-preserving (modk)-network such that all compact subsets of X are finite. But

X has not a σ-WHCP k-network.

Proof. Let $X = \{p\} \cup \{x_a : a < \omega_1\}$, where $p \notin \{x_a : a < \omega_1\}$. We define the topology for X as follows: a subset V of X is open if and only if X-V either is countable or includes p. Then we can easily see that X is a regular, Hausdorff space. Put $\mathcal{P} = \{\{p, x_a\} : a < \omega_1\}$. Then $\mathcal{P}$ is a closure-preserving closed cover of X because any subset of X missing p is open. Thus $\mathcal{P}$ is a closure-preserving (modk)-network for X with respect to the covering $\mathcal{P}$ of X by compact sets. If K is an infinite subset of X, put $F = K - \{p\}$. Then $\{X-F\} \cup \{\{x\} : x \in F\}$ is an open cover of K which has not any finite subcover of K. So all compact subsets of X are finite. Since X is Lindelöf, X has not a σ-WHCP k-network by Lemma 1.

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