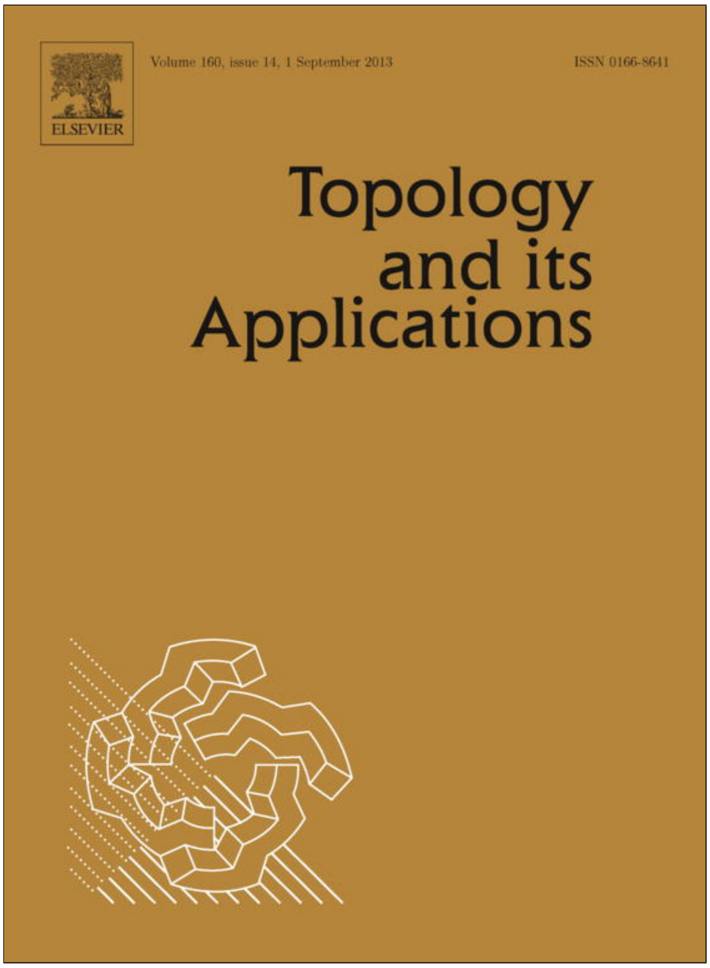
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Factorization properties of paratopological groups

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ABSTRACT

In this article we continue the study of $\mathbb{R}$-factorizability in paratopological groups. It is shown that: (1) all concepts of $\mathbb{R}$-factorizability in paratopological groups coincide; (2) a Tychonoff paratopological group G is $\mathbb{R}$-factorizable if and only if it is totally ω-narrow and has property ω-QU; (3) every subgroup of a T_1 paratopological group G is $\mathbb{R}$-factorizable provided that the topological group G^* associated to G is a Lindelöf Σ-space, i.e., G is a *totally Lindelöf Σ-space*; (4) if $\Pi = \prod_{i\in I} G_i$ is a product of T_1 paratopological groups which are totally Lindelöf Σ-spaces, then each dense subgroup of Π is $\mathbb{R}$-factorizable. These results answer in the affirmative several questions posed earlier by M. Sanchis and M. Tkachenko and by S. Lin and L.-H. Xie.

1. Introduction

A *paratopological* (*semitopological*) group is a group with a topology such that multiplication on the group is jointly (separately) continuous. If in addition inversion on the group is continuous, then it is called a *topological* (*quasitopological*) group.

For every continuous real-valued function f on a compact topological group G, one can find a continuous homomorphism $p : G \to L$ onto a second-countable topological group L and a continuous real-valued function h on L such that $f = h \circ p$ (see [7, Example 37]). The conclusion remains valid for pseudocompact topological groups, a result due to W.W. Comfort and K.A. Ross [3]. These facts motivated the third listed author to introduce $\mathbb{R}$-*factorizable* groups in [15] as the topological groups G with the property that every continuous real-valued function on G can be factorized through a continuous homomorphism onto a second-countable topological group. The class of $\mathbb{R}$-factorizable groups is unexpectedly wide. For example,

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it contains arbitrary subgroups of σ-compact (and even Lindelöf Σ-) groups, topological products of Lindelöf Σ-groups, and their dense subgroups [14]. For other properties of this class of topological groups, the reader is referred to [16,17].

Similarly to the case of topological groups, M. Sanchis and M. Tkachenko introduced in [11] the classes of $\mathbb{R}_i$-factorizable paratopological groups, for $i \in \{1, 2, 3, 3.5\}$. The need in the use of four different subscripts was due to the fact that the classes of T_1, Hausdorff, regular and, possibly, completely regular paratopological groups are all distinct, while T_0 topological groups are completely regular. As in the case of topological groups, the classes of $\mathbb{R}_i$-factorizable paratopological groups are very wide. For example, it was proved in [11] that every Hausdorff (regular) Lindelöf totally ω-narrow paratopological group is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable), and that every subgroup of a Hausdorff (regular) σ-compact paratopological group is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable). Also, it was recently shown in [12] that for every continuous real-valued function f on a *feebly compact* paratopological group G (no separation requirement on G is imposed), one can find a continuous homomorphism $\pi : G \to H$ onto a compact metrizable *topological group* H and a continuous real-valued function h on H such that $f = h \circ \pi$. As usual, we call a space X feebly compact if every locally finite family of open sets in X is finite.

The following question was posed by M. Sanchis and M. Tkachenko in [11].

Question 1.1. ([11, Question 5.4]) Suppose that H is a Hausdorff paratopological group such that the associated topological group H^* is a Lindelöf Σ-space. Is every subgroup of H $\mathbb{R}_2$-factorizable?

We answer this question affirmatively in Theorem 4.8 even if H satisfies only the T_1 separation axiom.

For the further study of $\mathbb{R}$-factorizable topological groups, L.-H. Xie and S. Lin generalized the concept of uniform continuity of real-valued functions on topological groups. Modifying *property* U defined in [5], they introduced *property* ω-U and established that a topological group is $\mathbb{R}$-factorizable if and only if it is ω-narrow and has property ω-U (see [22, Theorem 4.9]). Recently, with the aim to study open homomorphic images of $\mathbb{R}_i$-factorizable paratopological groups, L.-H. Xie and S. Lin [23] extended property ω-U to paratopological groups. They proved that if G is a completely regular $\mathbb{R}_2$-factorizable ($\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ is a continuous open homomorphism onto a paratopological group K satisfying $Hs(K) \leqslant \omega$ ($Ir(K) \leqslant \omega$), then K is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable). Here $Hs(G)$ and $Ir(G)$ stand, respectively, for the Hausdorff number and the index of regularity of the paratopological group G (see the definitions in [18]). We show in Proposition 3.20 that both restrictions $Hs(K) \leqslant \omega$ and $Ir(K) \leqslant \omega$ on the group K in [23] can be dropped.

It was also proved in [23] that every dense subgroup of a topological product of regular paratopological groups which are Lindelöf Σ-spaces is $\mathbb{R}_3$-factorizable and the following question was posed:

Question 1.2. ([23, Question 6.1]) Are dense subgroups of topological products of Hausdorff σ-compact paratopological groups $\mathbb{R}_2$-factorizable?

We give the positive answer to this question in Corollary 4.15 even in the case when the factors are T_1-spaces.

The article is organized as follows. In Section 3 we modify slightly the original definition of $\mathbb{R}_i$-factorizability given in [11] by eliminating the separation restrictions on a paratopological group G (but keeping these restrictions for second-countable continuous homomorphic images of G). Having a recourse to [21], we show that *all* variants of $\mathbb{R}$-factorizability in paratopological groups are equivalent to its weakest form, when the second-countable continuous homomorphic images of a given paratopological group are not assumed to satisfy any separation axiom.

Making use of property ω-QU (a form of property ω-U designed for paratopological groups), we characterize Tychonoff $\mathbb{R}$-factorizable paratopological groups. It is proved in Theorem 3.14 that every totally

ω-narrow paratopological group with property ω-QU is $\mathbb{R}$-factorizable. It follows from Theorem 3.14 that every quotient of a Tychonoff $\mathbb{R}$-factorizable paratopological group is $\mathbb{R}$-factorizable as well (see Proposition 3.20). In Theorem 3.21 we establish that a Tychonoff paratopological group G is $\mathbb{R}$-factorizable if and only if it is totally ω-narrow and has property ω-QU.

Our aim in Section 4 is to study $\mathbb{R}$-factorizability in σ-compact paratopological groups satisfying the T_1 separation axiom. In fact, we work in the wider class of paratopological groups which are Lindelöf Σ-spaces and which are called *$L\Sigma$-groups* for brevity. It is shown in Theorem 4.8 that every subgroup of a *totally $L\Sigma$-group* satisfying the T_1 separation axiom is $\mathbb{R}$-factorizable (the definition of 'total' is given in Section 2). Theorem 4.8 implies that every subgroup of a σ-compact T_1 paratopological group is $\mathbb{R}$-factorizable (see Corollary 4.10). Further, if $G = \prod_{i\in I} G_i$ is the product of a family of σ-compact paratopological groups satisfying the T_1 separation axiom, then every dense subgroup of G is $\mathbb{R}$-factorizable (see Corollary 4.15). These results answer Questions 1.1 and 1.2 in the affirmative.

Finally, in Section 5, we formulate several open problems.

2. Notation and preliminary facts

The spaces we consider are not assumed to satisfy any separation axiom, unless the otherwise is stated explicitly. Further, T_3 and $T_{3.5}$ do not include T_1, while 'regular' and 'completely regular' mean $T_3 + T_1$ and $T_{3.5} + T_1$, respectively.

By $l(X)$ we denote the Lindelöf number of a space X.

Given a paratopological group G, we denote by $ib(G)$ the minimal cardinal number $\kappa \geqslant \omega$ such that for every neighborhood U of the identity in G, there exists a subset F of G such that $|F| \leqslant \kappa$ and $FU = G = UF$. The cardinal $ib(G)$ is called the *index of narrowness* of the group G [2, Section 5.2]. If $ib(G) \leqslant \omega$, we say that G is *ω-narrow*.

For a paratopological group G with topology τ, one defines the *conjugate topology* τ^{-1} on G by $\tau^{-1} = \{U^{-1}\colon U \in \tau\}$. Then $G' = (G, \tau^{-1})$ is also a paratopological group, and the inversion $x \to x^{-1}$ is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on G, and we call $G^* = (G, \tau^*)$ the topological group *associated* to G. It is easy to see that the family $\{U \cap U^{-1}\colon e \in U \in \tau\}$ forms a local base at the neutral element e of the group G^*.

For further references, we collect several basic facts here, some old and some folklore, about the interaction of G and G^*. A simple verification of them is left to the reader.

Proposition 2.1. *Let G be a paratopological group.*

(1) *If G satisfies the T_0 separation axiom, then G^* is a Hausdorff topological group.*
(2) *If H is an arbitrary subgroup of G, then H^* is topologically isomorphic, under the identity mapping, to a subgroup of G^*.*
(3) *Let $f : G \to K$ be a continuous homomorphism of paratopological groups. Then the mapping $f^* : G^* \to K^*$, which coincides pointwise with f, is a continuous homomorphism of topological groups.*
(4) *For an arbitrary product $\Pi = \prod_{i\in I} G_i$ of paratopological groups, the identity mapping of Π^* onto $\prod_{i\in I}(G_i)^*$ is a topological isomorphism of topological groups.*

The following important fact was established in [1, Lemma 2.2].

Lemma 2.2. *Let G be a paratopological group with topology τ and τ^{-1} be the conjugate topology on G. Then the topological group G^* associated to G is topologically isomorphic to the diagonal $\Delta = \{(x, x)\colon x \in G\}$ considered as a subspace of $(G, \tau) \times (G, \tau^{-1})$. If G satisfies the T_1 separation axiom, then Δ is closed in $(G, \tau) \times (G, \tau^{-1})$.*

Let $\mathcal{P}$ be a (topological) property. According to [11, Definition 3.1], a paratopological group G is called *totally* $\mathcal{P}$ if the associated topological group G^* has property $\mathcal{P}$. Thus we say that a paratopological group G is *totally ω-narrow* if the associated topological group G^* is ω-narrow.

Following [9], we define the *symmetry number* of a T_1 paratopological group G, denoted by $Sm(G)$, as is the minimum cardinal number κ such that for every neighborhood U of the identity e in G, there exists a family γ of neighborhoods of e with $1 \leqslant |\gamma| \leqslant \kappa$ such that $\bigcap \gamma \subset U^{-1}$. Notice that a T_1 paratopological group G is a topological group iff $Sm(G) = 1$. It is worth mentioning that the symmetry number was called *weak Hausdorff number* in [24].

The next theorem shows the importance of the symmetry number (see [9, Theorem 2.19]):

Theorem 2.3. *A T_1 paratopological group G is topologically isomorphic to a subgroup of a product of second-countable T_1 paratopological groups iff G is totally ω-narrow and satisfies $Sm(G) \leqslant \omega$.*

The case of embeddings into products of regular second-countable paratopological groups was considered in [18]. It was shown in [18, Theorem 3.8] that a regular paratopological group G is topologically isomorphic to a subgroup of a product of regular second-countable paratopological groups if and only if G is totally ω-narrow and the *index of regularity* of G is countable. Recently, I. Sánchez proved in [9] that every regular totally ω-narrow paratopological group has countable index of regularity. Hence Theorem 3.8 of [18] can be given the following elegant form:

Theorem 2.4. *A regular paratopological group G is topologically isomorphic to a subgroup of a product of regular second-countable paratopological groups if and only if G is totally ω-narrow.*

We will use the following fact established recently in [9, Proposition 2.4] and, independently, in [24, Corollary 2.4].

Lemma 2.5. *Every T_1 paratopological group G satisfies $Sm(G) \leqslant l(G)$. In particular, if G is Lindelöf, then the symmetry number of G is countable.*

It follows from [10, Proposition 3.8] that every totally ω-narrow paratopological group is ω-balanced,[3] while [10, Proposition 3.5] implies that every first-countable totally ω-narrow paratopological group is second-countable. Therefore, Theorems 2.3 and 2.4 on isomorphic embeddings of paratopological groups into products can be given the following equivalent form:

Lemma 2.6. *Let G be a totally ω-narrow paratopological group.*

(1) *If G is a T_1-space with $Sm(G) \leqslant \omega$, then for every open neighborhood U of the identity in G, there exists a continuous homomorphism π of G onto a second-countable T_1 paratopological group H such that $\pi^{-1}(V) \subset U$, for some open neighborhood V of the identity in H.*
(2) *If G is regular, then for every open neighborhood U of the identity in G, there exists a continuous homomorphism π of G onto a regular second-countable paratopological group H such that $\pi^{-1}(V) \subset U$, for some open neighborhood V of the identity in H.*

A subset U of a space X is called *regular open* if $U = \mathrm{Int}(\overline{U})$. Similarly, a subset F of a space X is called *regular closed* if $F = \overline{\mathrm{Int}(F)}$. Given a space (X, τ), denote by τ' the topology on X whose base consists of regular open subsets of (X, τ). The space (X, τ') is said to be the *semiregularization* of (X, τ) and is

[3] A paratopological group G is *ω-balanced* if for every neighborhood U of the identity e in G, there exists a countable family γ of open neighborhoods of e in G such that for each $x \in G$ one can find $V \in \gamma$ satisfying $xVx^{-1} \subset U$ (see [2, Section 3.4]).

denoted by X_{sr}. It is easy to see that $\tau' \subset \tau$ and that the spaces (X, τ) and (X, τ') have the same regular open and regular closed subsets.

The operation of semiregularization was defined by M. Stone in [13] and studied by M. Katetov [4].

The following very useful result was proved by Ravsky in [8] (see also [20, Theorem 2.1]):

Theorem 2.7. *Let G be an arbitrary paratopological group. Then the space G_{sr} carrying the same group structure is a T_3 paratopological group. If G is Hausdorff, then G_{sr} is a regular paratopological group.*

3. Characterizing $\mathbb{R}$-factorizable paratopological groups

The classes of $\mathbb{R}_i$-factorizable paratopological groups, for $i = 1, 2, 3, 3.5$, were introduced in [11] as natural extensions of the class of $\mathbb{R}$-factorizable topological groups. Recently, using property ω-QU, L.-H. Xie and S. Lin gave some characterizations of $\mathbb{R}_2$- and $\mathbb{R}_3$-factorizable paratopological groups. In this section we show that all concepts of $\mathbb{R}_i$-factorizability for paratopological groups coincide, for $i = 1, 2, 3, 3.5$, when the separation restrictions on G are eliminated from the definition in [11]. Then we characterize the class of Tychonoff $\mathbb{R}$-factorizable paratopological groups.

Several results on $\mathbb{R}$-factorizability in paratopological groups show that the original definition of this concept given in [11] is somewhat restrictive. It turns out that $\mathbb{R}_3$-factorizability of a given paratopological group G can sometimes be established without the requirement that G is regular (see [12, Theorem 5] and our Corollaries 4.9 and 4.10). Therefore, we prefer to change the definition in [11] by eliminating the separation restrictions on G:

Definition 3.1. A paratopological group G is $\mathbb{R}_0$-factorizable ($\mathbb{R}_i$-factorizable, for $i = 1, 2, 3, 3.5$) if for every continuous real-valued function f on G, one can find a continuous homomorphism $\pi : G \to H$ onto a second-countable paratopological group H satisfying the T_0 (resp., $T_i + T_1$) separation axiom and a continuous real-valued function h on H such that $f = h \circ \pi$. If we do not impose any separation restriction on H, we obtain the concept of $\mathbb{R}$-factorizability.

In fact, the above definition has already been used in [12, Section 3].

Remark 3.2. From Definition 3.1 one can deduce the following:

(a) $\mathbb{R}_i$-factorizability implies $\mathbb{R}_j$-factorizability, whenever $0 \leqslant j < i \leqslant 3.5$. Also, $\mathbb{R}_0$-factorizability implies $\mathbb{R}$-factorizability.
(b) Clearly, every second-countable regular space is completely regular. Hence a paratopological group is $\mathbb{R}_3$-factorizable iff it is $\mathbb{R}_{3.5}$-factorizable. This is why we shall deal with $\mathbb{R}_3$-factorizability rather than $\mathbb{R}_{3.5}$-factorizability.
(c) We will see in Proposition 3.7 that $\mathbb{R}_2$-factorizability implies $\mathbb{R}_3$-factorizability. Hence the concepts of $\mathbb{R}_2$-, $\mathbb{R}_3$-, and $\mathbb{R}_{3.5}$-factorizability coincide.
(d) According to Proposition 3.4, the classes of $\mathbb{R}$-, $\mathbb{R}_0$-, $\mathbb{R}_1$-, and $\mathbb{R}_2$-factorizable paratopological groups coincide. Hence Definition 3.1 introduces only one class of paratopological groups.

To show that all concepts of $\mathbb{R}$- and $\mathbb{R}_i$-factorizability (for $i = 0, 1, 2, 3$) in paratopological groups coincide, we need the following result proved in [21]:

Theorem 3.3. *Let H be an arbitrary paratopological (semitopological) group. Then there exists a continuous open homomorphism $\pi : H \to T_2(H)$ onto a Hausdorff paratopological (semitopological) group $T_2(H)$ such that for every continuous mapping $f : H \to X$ to a Hausdorff space X, one can find a continuous mapping $h : T_2(H) \to X$ satisfying $f = h \circ \pi$.*

Proposition 3.4. *Every* $\mathbb{R}$*-factorizable paratopological group is* $\mathbb{R}_2$*-factorizable. Hence* $\mathbb{R}$*-,* $\mathbb{R}_0$*-,* $\mathbb{R}_1$*-, and* $\mathbb{R}_2$*-factorizability in paratopological groups coincide.*

Proof. Let f be a continuous real-valued function on an $\mathbb{R}$-factorizable paratopological group G. Then we can find a continuous homomorphism p of G onto a second-countable paratopological group H and a continuous real-valued function g on H such that $f = g \circ p$. By Theorem 3.3, there exist an open continuous homomorphism $\pi : H \to T_2(H)$ onto a Hausdorff paratopological group $T_2(H)$ and a continuous real-valued function h on $T_2(H)$ such that $g = h \circ \pi$. Since the homomorphism π is open, the group $T_2(H)$ is also second-countable. Then $\varphi = \pi \circ p$ is a continuous homomorphism of G onto a second-countable Hausdorff paratopological group and $f = h \circ \varphi$. Hence G is $\mathbb{R}_2$-factorizable. □

It remains to show that $\mathbb{R}_2$-factorizability implies $\mathbb{R}_3$-factorizability in paratopological groups. This requires two simple lemmas.

Lemma 3.5. *Let* $f : X \to Y$ *be a continuous mapping of* X *to a regular space* Y*. Then* f *remains continuous as a mapping of the semiregularization* X_{sr} *of* X *to* Y*.*

Proof. Take a point $x \in X$ and a neighborhood U of $f(x)$ in Y. Let V be an open neighborhood of $f(x)$ such that $\overline{V} \subset U$. Since f is continuous on X, we can find an open neighborhood O of x in X such that $f(O) \subset V$. Hence $f(\overline{O}) \subset \overline{V} \subset U$ and, consequently, $f(\operatorname{Int} \overline{O}) \subset f(\overline{O}) \subset U$. Since $\operatorname{Int} \overline{O}$ is an open neighborhood of x in X_{sr}, we see that f is continuous on X_{sr}. □

Lemma 3.6. *If* X *is a second-countable space, then so is* X_{sr}*.*

Proof. Let $\mathcal{B}$ be a countable base for X. A direct verification shows that the countable family $\mathcal{C} = \{\operatorname{Int} \overline{U}\colon\ U \in \mathcal{B}\}$ is a base for X_{sr}. □

Proposition 3.7. *Every* $\mathbb{R}_2$*-factorizable paratopological group* G *is* $\mathbb{R}_3$*-factorizable.*

Proof. Let f be a continuous real-valued function on G. Then one can find a continuous homomorphism $p : G \to K$ onto a Hausdorff paratopological group K of countable weight and a continuous real-valued function g on K such that $f = g \circ p$.

Denote by K_{sr} the semiregularization of K. Since K is a Hausdorff paratopological group, Theorem 2.7 implies that K_{sr} is a regular paratopological group. By Lemma 3.6, K_{sr} has a countable base. Denote by g_{sr} the function g considered as a mapping of K_{sr} to the real line. Lemma 3.5 implies that g_{sr} is continuous on K_{sr}. Let also i_K be the identity isomorphism of K onto K_{sr}. Since $f = g_{sr} \circ i_K \circ p$, we conclude that G is $\mathbb{R}_3$-factorizable. □

Combining Propositions 3.4 and 3.7, we obtain the following result:

Theorem 3.8. *Every* $\mathbb{R}$*-factorizable paratopological group is* $\mathbb{R}_3$*-factorizable. Hence the concepts of* $\mathbb{R}$*-,* $\mathbb{R}_0$*-,* $\mathbb{R}_1$*,* $\mathbb{R}_2$*-, and* $\mathbb{R}_3$*-factorizability coincide in the class of paratopological groups.*

Also, we find it useful to formulate the following fact obtained by a simple combination of Theorems 2.7, 3.3, and Lemma 3.5:

Proposition 3.9. *Let* $f : G \to Y$ *be a continuous mapping of a paratopological group* G *to a regular space* Y*. Then one can find a continuous homomorphism* $p : G \to H$ *of* G *onto a regular paratopological group* H *and a continuous mapping* $h : H \to Y$ *such that* $f = h \circ p$*.*

Theorem 3.8 enables us to avoid the use of the terms '$\mathbb{R}_i$-factorizability' for $i = 0, 1, 2, 3, 3.5$ by replacing them to the shorter (but equivalent) term '$\mathbb{R}$-factorizability'. Furthermore, Theorem 3.8 shows that all results on $\mathbb{R}_i$-factorizable paratopological groups proved in [11,12,23] for $i = 0, 1, 2, 3$ can be equivalently reformulated for the class of $\mathbb{R}$-factorizable paratopological groups.

The next result is a reformulation of this kind. It is obtained as a combination of [11, Proposition 3.5] and our Theorem 3.8. We have to mention that the word 'Tychonoff' was erroneously omitted in the conditions on the paratopological group G in Proposition 3.5 of [11].

Proposition 3.10. *Every Tychonoff $\mathbb{R}$-factorizable paratopological group G is totally ω-narrow.*

According to [23], a real-valued function f on a paratopological group G is called *left* (*right*) *ω-quasi-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U}$ of open neighborhoods of the identity in G with the property that for every $x \in G$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in xU$ (resp., $y \in Ux$).

A real-valued function f on a paratopological group G is *ω-quasi-uniformly continuous* [23] if f is both left and right ω-quasi-uniformly continuous.

The following lemma is immediate from the definition of left (right) ω-quasi-uniform continuity (see [23, Proposition 4.4]).

Lemma 3.11. *Let f be a real-valued function defined on a paratopological group G. The following conditions are equivalent*:

(1) *f is left* (*right*) *ω-quasi-uniformly continuous*;
(2) *there exists a countable family $\mathcal{U}$ of open neighborhoods of the identity in G with the property that for every point $x \in G$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in xU$ (resp., $y \in Ux$).*

The next result extends Proposition 3.9 to ω-quasi-uniformly continuous functions.

Lemma 3.12. *Let f be a left* (*right*) *ω-quasi-uniformly continuous real-valued function on a paratopological group G. Then one can find a continuous homomorphism $p : G \to K$ onto a regular paratopological group K and a left* (*right*) *ω-quasi-uniformly continuous function h on K such that $f = h \circ p$.*

Proof. Suppose that f is left ω-quasi-uniformly continuous. We do the job in two steps. First, applying Theorem 3.3, we find a continuous open homomorphism $\pi : G \to H$ of G onto a Hausdorff paratopological group H and a continuous function g on H such that $f = g \circ \pi$. Since the homomorphism π is open, the function g is left ω-quasi-uniformly continuous and, hence, continuous.

Let H_{sr} be the semiregularization of H. Then H_{sr} is a regular paratopological group, by Theorem 2.7. It follows from Lemma 3.5 that g remains continuous when considered as a function on H_{sr}. Denote by i the identity mapping of H onto H_{sr}. Then $p = i \circ \pi$ is a continuous homomorphism of G onto $K = H_{sr}$ and $f = h \circ p$, where $h : H_{sr} \to \mathbb{R}$ is the function which coincides with g pointwise. It remains to verify that h is left ω-quasi-uniformly continuous on K.

Let $\mathcal{U}$ be a countable family of open neighborhoods of the neutral element in H witnessing that the function g is left ω-quasi-uniformly continuous. Denote by $\mathcal{V}$ the family $\{\operatorname{Int} \overline{U}\colon U \in \mathcal{U}\}$. If $x \in H$ and $\epsilon > 0$, there exists $U \in \mathcal{U}$ such that $|g(x) - g(y)| < \epsilon$ for each $y \in xU$. Since g is continuous, it follows that $|g(x) - g(xz)| \leqslant \epsilon$ for each $z \in \overline{U}$ and, hence, the inequality $|h(x) - h(xz)| \leqslant \epsilon$ holds for each $z \in \operatorname{Int} \overline{U}$. Since $\operatorname{Int} \overline{U} \in \mathcal{V}$, we conclude that h is left ω-quasi-uniformly continuous on $H_{sr} = K$. The argument in the case of a right ω-quasi-uniformly continuous function f is similar. $\square$

The following fact is a sharper and more general version of Lemma 4.13 in [23].

Lemma 3.13. *Let G be a totally ω-narrow paratopological group and $f : G \to \mathbb{R}$ be either a left or right ω-quasi-uniformly continuous function. Then one can find a continuous homomorphism $\pi : G \to L$ onto a regular second-countable paratopological group L and a continuous function $h : L \to \mathbb{R}$ such that $f = h \circ \pi$.*

Proof. Suppose that a function $f : G \to \mathbb{R}$ is left ω-quasi-uniformly continuous. According to Lemma 3.12, we can assume without loss of generality that G is regular. Indeed, take a continuous homomorphism $p : G \to K$ onto a regular paratopological group K and a left ω-quasi-uniformly continuous function g on K such that $f = g \circ p$. By item (3) of Proposition 2.1, K is totally ω-narrow. Therefore, we can replace G with K and f with g, respectively.

By Lemma 3.11, there exists a countable family $\mathcal{U}$ of open neighborhoods of the identity in G such that for any $\varepsilon > 0$ and any point $x \in G$, there exists $U \in \mathcal{U}$ satisfying $|f(x) - f(xu)| < \varepsilon$ for all $u \in U$. According to (2) of Lemma 2.6, one can find a continuous homomorphism $\pi_U : G \to H_U$ onto a regular second-countable paratopological group H_U such that $\pi_U^{-1}(V) \subset U$ for some open neighborhood V of the identity in H_U. Let $\pi = \Delta_{U \in \mathcal{U}} \pi_U$ be the diagonal product of the family $\{\pi_U \colon U \in \mathcal{U}\}$. It is clear that $L = \pi(G)$ is a regular second-countable paratopological group since the product space $\prod_{U \in \mathcal{U}} H_U$ is regular and second-countable.

Claim. *$f(x_1) = f(x_2)$ for all $x_1, x_2 \in G$ satisfying $\pi(x_1) = \pi(x_2)$.*

Indeed, assume the contrary and choose $x_1, x_2 \in G$ and $\varepsilon > 0$ such that

$$\pi(x_1) = \pi(x_2) \quad \text{and} \quad \bigl|f(x_1) - f(x_2)\bigr| \geqslant \varepsilon.$$

By our choice of $\mathcal{U}$, for x_2 and ε, there exists $U \in \mathcal{U}$ such that $|f(x_2) - f(x_2 u)| < \varepsilon$ for all $u \in U$, which is equivalent to $f(x_2 U) \subset (f(x_2) - \varepsilon, f(x_2) + \varepsilon)$. Take an open neighborhood V of the identity in H_U such that $\pi_U^{-1}(V) \subset U$ and put $y = \pi_U(x_1)$. Then $y = \pi_U(x_2)$ since $\pi(x_1) = \pi(x_2)$, and

$$x_1 \in \pi_U^{-1}(yV) = x_2 \pi_U^{-1}(V) \subset x_2 U.$$

This in turn implies that

$$f(x_1) \in f(x_2 U) \subset \bigl(f(x_2) - \varepsilon, f(x_2) + \varepsilon\bigr),$$

whence $|f(x_1) - f(x_2)| < \varepsilon$. This contradicts our choice of the elements x_1, x_2 and proves the claim.

It follows from the above claim that there exists a real-valued function h on $L = \pi(G)$ such that $f = h \circ \pi$. It remains to prove that h is continuous.

Take any $\varepsilon > 0$, $y \in \pi(G)$, and pick a point $x \in G$ such that $y = \pi(x)$. Since $f = h \circ \pi$, our choice of $\mathcal{U}$ implies that there exists $U \in \mathcal{U}$ such that

$$f(xU) \subset \bigl(f(x) - \varepsilon, f(x) + \varepsilon\bigr) = \bigl(h(y) - \varepsilon, h(y) + \varepsilon\bigr). \tag{1}$$

By the definition of π_U, there is an open neighborhood V of the identity in H_U such that $\pi_U^{-1}(V) \subset U$. Put

$$O = \pi(G) \cap \Biggl(V \times \prod_{U' \in \mathcal{U} \setminus \{U\}} H_{U'}\Biggr).$$

Note that O is an open neighborhood of the identity in $\pi(G)$ and $\pi^{-1}(O) = \pi_U^{-1}(V) \subset U$. Therefore, it follows from (1) that

$$h(yO) \subset f\bigl(\pi^{-1}(yO)\bigr) = f\bigl(x\pi^{-1}(O)\bigr) \subset f(xU) \subset \bigl(h(y) - \varepsilon, h(y) + \varepsilon\bigr).$$

Hence the function h is continuous. This completes the proof in the case when f is left ω-quasi-uniformly continuous. The argument in the case of a right ω-quasi-uniformly continuous function f is similar. □

Following [23] we say that a paratopological group G has *property ω-QU* if all continuous real-valued functions on G are ω-quasi-uniformly continuous. The next theorem follows directly from Lemma 3.13.

Theorem 3.14. *Every totally ω-narrow paratopological group with property ω-QU is $\mathbb{R}$-factorizable.*

Corollary 3.15. *Every totally ω-narrow Lindelöf paratopological group G is $\mathbb{R}$-factorizable.*

Proof. By [23, Theorem 4.10], every Lindelöf paratopological group has property ω-QU. Hence the conclusion follows from Theorem 3.14. □

Corollary 3.16. *Every paratopological group with a countable network is $\mathbb{R}$-factorizable.*

Lemma 3.17. *Every $\mathbb{R}$-factorizable paratopological group has property ω-QU.*

Proof. Let $f : G \to \mathbb{R}$ be a continuous function on an $\mathbb{R}$-factorizable paratopological group G. Then we can find a continuous homomorphism $\pi : G \to K$ onto a second-countable paratopological group K and a continuous function h on K such that $f = h \circ \pi$. Let $\mathscr{B}$ be a countable local base at the identity of K. Put $\mathscr{U} = \{\pi^{-1}(U)\colon U \in \mathscr{B}\}$. One can easily verify that $\mathscr{U}$ is a countable family of open neighborhoods of the identity in G which has the property that for every point $x \in G$ and every $\varepsilon > 0$, there exists an element $U \in \mathcal{U}$ such that $|f(x) - f(xu)| < \varepsilon$ and $|f(x) - f(ux)| < \varepsilon$ for each $u \in U$. Hence Lemma 3.11 implies that f is ω-quasi-uniformly continuous. So G has property ω-QU. □

In fact, we can apply Lemma 3.17 to reformulate Theorem 3.14 as follows:

Theorem 3.18. *A totally ω-narrow paratopological group G is $\mathbb{R}$-factorizable if and only if it has property ω-QU.*

It is still an open problem whether every quotient of an $\mathbb{R}$-factorizable paratopological group is $\mathbb{R}$-factorizable (see [11, Problem 5.2]). We solve this problem in the classes of totally ω-narrow and Tychonoff $\mathbb{R}$-factorizable paratopological groups.

Proposition 3.19. *Every quotient of a totally ω-narrow $\mathbb{R}$-factorizable paratopological group G is $\mathbb{R}$-factorizable.*

Proof. Lemma 3.17 implies that G has property ω-QU. By [23, Lemma 5.2], property ω-QU is preserved under taking arbitrary quotients. Item (3) of Proposition 2.1 implies that total ω-narrowness is preserved by continuous surjective homomorphisms as well. Hence the required conclusion follows from Theorem 3.14. □

Proposition 3.20. *Every quotient of a Tychonoff $\mathbb{R}$-factorizable paratopological group G is $\mathbb{R}$-factorizable.*

Proof. It follows from Proposition 3.10 that the group G is totally ω-narrow. Hence the $\mathbb{R}$-factorizability of G follows from Proposition 3.19. □

Making use of property ω-QU we characterize Tychonoff $\mathbb{R}$-factorizable paratopological groups in the theorem below.

Theorem 3.21. *A Tychonoff paratopological group G is $\mathbb{R}$-factorizable if and only if it is totally ω-narrow and has property ω-QU.*

Proof. The sufficiency follows from Theorem 3.14. Conversely, suppose that G is a Tychonoff $\mathbb{R}$-factorizable paratopological group. Then G is totally ω-narrow by Proposition 3.10, and Lemma 3.17 implies that G has property ω-QU. □

Remark 3.22. Let $\mathbb{S}$ be the Sorgenfrey line and $\mathbb{Z}$ the group of integers, which is a closed subgroup of $\mathbb{S}$. Then the quotient paratopological group $\mathbb{T}_{Sor} = \mathbb{S}/\mathbb{Z}$ is algebraically the additive circle group whose local base at the neutral element $\bar{0}$ is generated by the half-open intervals $\{[0, \frac{1}{n})\colon n \in \mathbb{N}^+\}$. The paratopological group $\mathbb{T}_{Sor}$ is regular, hereditarily Lindelöf, hereditarily separable, but it is not totally ω-narrow. Hence Theorem 3.21 implies that $\mathbb{T}_{Sor}$ is not $\mathbb{R}$-factorizable (see also [11, Example 3.3]). Clearly, the Sorgenfrey line $\mathbb{S}$ has the same properties, but $\mathbb{T}_{Sor}$ is, in addition, *precompact*. The latter means that the group $\mathbb{T}_{Sor}$ can be covered by finitely many translates of any neighborhood of the identity. It should be noted, however, that precompact *topological* groups are $\mathbb{R}$-factorizable [2, Corollary 8.1.17].

4. $\mathbb{R}$-factorizability in totally $L\Sigma$-groups

In this section we consider $\mathbb{R}$-factorizability in σ-compact and totally $L\Sigma$-groups satisfying the T_1 separation axiom.

The following lemma plays an important role here. It extends [10, Corollary 3.13] to the case of T_1 paratopological groups.

Lemma 4.1. *Let G be a Lindelöf totally ω-narrow T_1 paratopological group with neutral element e. Then, for every G_δ-set P in G with $e \in P$, there exists a closed invariant subgroup N of G such that $N \subset P$ and the quotient paratopological group G/N has countable pseudocharacter.*

Proof. Let $P = \bigcap_{i\in\omega} U_i$, where each U_i is an open neighborhood of e in G. Since G is Lindelöf, it follows from Lemma 2.5 that $Sm(G) \leqslant \omega$. Then item (1) of Lemma 2.6 implies that for each $i \in \omega$, there exists a continuous homomorphism $\pi_i : G \to H_i$ onto a second-countable T_1 paratopological group H_i such that $\pi_i^{-1}(V_i) \subset U_i$, for some open neighborhood V_i of the neutral element in H_i.

Let $\pi = \Delta_{i\in\omega}\pi_i$ be the diagonal product of the family $\{\pi_i\colon i \in \omega\}$. Put $N = \pi^{-1}(e')$, where e' is the neutral element of $H = \prod_{i\in\omega} H_i$. Clearly, N is closed in G and

$$N = \bigcap_{i\in\omega} \pi_i^{-1}(e_i) \subset \bigcap_{i\in\omega} \pi_i^{-1}(V_i) \subset \bigcap_{i\in\omega} U_i = P,$$

where e_i is the neutral element of H_i for each $i \in \omega$. It is easy to see that the quotient paratopological group G/N is T_1 and has countable pseudocharacter, since the canonical one-to-one mapping $id : G/N \to \pi(G)$ is continuous and the paratopological groups H and $\pi(G) \subset H$ are T_1-spaces of countable weight. □

In what follows we work with a paratopological group G such that either G or the associated topological group G^* is a Lindelöf Σ-space. Usually the Lindelöf Σ-spaces are assumed to be Tychonoff (see [2, Section 5.3]). This is why we consider only paratopological groups satisfying the T_1 separation axiom in this section—then the topological group G^* associated to G is Tychonoff and, by Lemma 2.2, is topologically isomorphic to a closed subgroup of $G \times G'$. However, one can use the original definition of *Σ-spaces* given by Nagami in [6], where the spaces are assumed to be Hausdorff.

An equivalent reformulation of the definition of Σ-spaces in [6] is as follows. It is said that X is a *Σ-space* if there are two coverings of X by closed sets, say, $\mathcal{C}$ and $\mathcal{F}$ such that $\mathcal{C}$ is σ-locally finite, $\mathcal{F}$ consists of countably compact sets, and for every $F \in \mathcal{F}$ and every open neighborhood U of F in X, one can find $C \in \mathcal{C}$ satisfying $F \subset C \subset U$. Needless to say that in the class of regular Lindelöf spaces, the Σ-spaces in the sense of Nagami are exactly the Lindelöf Σ-spaces considered in [2]. Hence we can deal with Lindelöf Σ-spaces which satisfy the Hausdorff separation axiom only.

It is clear that, for a Lindelöf Σ-space X, the corresponding covering $\mathcal{C}$ of X must be countable and the elements of the covering $\mathcal{F}$ must be compact. Similarly to [6, Theorem 3.13], one can show that any product of countably many Lindelöf Σ-spaces is again a Lindelöf Σ-space.

A space X is called *ω-cellular* or, in symbols, $cel_\omega(X) \leqslant \omega$, if every family γ of G_δ-sets in X contains a countable subfamily λ such that the $\bigcup \lambda$ is dense in $\bigcup \gamma$. We start with a lemma that generalizes a similar result proved in [10, Lemma 4.1] for Hausdorff totally $L\Sigma$-groups. Let us recall that the term '$L\Sigma$-group' refers to a paratopological group.

Lemma 4.2. *Let H be a totally $L\Sigma$-group satisfying the T_1 separation axiom. Then*:

(1) *the space H is ω-cellular*;
(2) *if γ is a countable family of closed G_δ-sets in H, then there exists a closed invariant subgroup N of H such that the quotient paratopological group H/N has a countable network and $F = \pi^{-1}(\pi(F))$, for each $F \in \gamma$, where $\pi : H \to H/N$ is the quotient homomorphism.*

Proof. Since H is T_1 and a totally $L\Sigma$-group, it follows from [2, Theorem 5.3.18] that the associated topological group H^* is Tychonoff and ω-cellular. Hence H is ω-cellular as a continuous image of H^*.

It remains to deduce (2) of the lemma. Let $\mathcal{N}$ be the family of closed invariant subgroups of H such that for each $N \in \mathcal{N}$, the quotient paratopological group H/N has countable pseudocharacter. One easily verifies that the family $\mathcal{N}$ is closed under countable intersections. Therefore, it suffices to prove (2) in the special case when γ contains a single element, say, F.

For every $x \in F$, $x^{-1}F$ is a G_δ-set in H which contains the identity e of H, so by Lemma 4.1, we can find an element $N_x \in \mathcal{N}$ such that $N_x \subset x^{-1}F$. The family $\{xN_x\colon x \in F\}$ covers F, and since the space H is ω-cellular, there exists a countable set $C \subset F$ such that $B = \bigcup_{x\in C} xN_x$ is dense in F. Then $N = \bigcap_{x\in C} N_x \in \mathcal{N}$. We claim that $F = \pi^{-1}(\pi(F))$, where $\pi : H \to H/N$ is the quotient homomorphism.

Indeed, take any $x \in C$. Since N_x is a subgroup of H and $N \subset N_x$, we have that $N_x = \pi^{-1}(\pi(N_x))$ and, hence, $xN_x = \pi^{-1}(\pi(xN_x))$. In its turn, this implies that $B = \pi^{-1}(\pi(B))$. Since the mapping π is open and B is dense in the closed set F, it follows that $F = \pi^{-1}(\pi(F))$.

Finally, we have to verify that the quotient paratopological group H/N has a countable network. To this end, consider the quotient topological group H^*/N^*. Clearly, the identity isomorphism $\varphi : H^*/N^* \to H/N$ is continuous. The pseudocharacter of H/N is countable by our choice of N, so the pseudocharacter of H^*/N^* is countable as well. Since H^* and H^*/N^* are Lindelöf Σ-spaces, it follows from [2, Corollary 5.3.25] that H^*/N^* has a countable network. Thus H/N has a countable network as a continuous image of H^*/N^*. □

The following lemma is evident.

Lemma 4.3. *Every T_3-space with a countable network has a countable closed network.*

According to [10, Theorem 4.2], the closure of the union of an arbitrary family of G_δ-sets in a regular $L\Sigma$-group is again a G_δ-set. The same conclusion is valid for Hausdorff σ-compact paratopological groups [10, Theorem 4.4]. The conclusion of our next result is weaker, but it holds for all $L\Sigma$-groups which are T_1-spaces:

Proposition 4.4. *Let H be a totally $L\Sigma$-group satisfying the T_1 separation axiom. Then the closure of every open subset of H is a G_δ-set.*

Proof. Let $\mathcal{N}$ be the family of closed invariant subgroups N of H such that the quotient paratopological group H/N has countable pseudocharacter. Since H is a totally $L\Sigma$-group, it is Lindelöf and totally ω-narrow. Take a non-empty open set U in H. For any $x \in U$, $x^{-1}U$ is an open set in H containing the identity of H. Making use of Lemma 4.1, we can find a closed invariant subgroup $N_x \in \mathcal{N}$ such that $N_x \subset x^{-1}U$. Clearly, the family $\{xN_x\colon x \in U\}$ covers U. According to (1) of Lemma 4.2, one can find a countable subset $C \subset U$ such that $B = \bigcup_{x\in C} xN_x$ is dense in U. Since xN_x is a closed G_δ-set in H for each $x \in U$, it follows from (2) of Lemma 4.2 that there exists a closed invariant subgroup N of H such that the quotient paratopological group H/N has a countable network and $xN_x = \pi^{-1}(\pi(xN_x))$, for each $x \in C$, where $\pi : H \to H/N$ is the quotient homomorphism. Thus $B = \pi^{-1}(\pi(B))$ and $\pi(B)$ is dense $\pi(U)$. Since π is an open mapping, the equalities

$$\overline{U} = \overline{B} = \overline{\pi^{-1}(\pi(B))} = \pi^{-1}\big(\overline{\pi(B)}\big) = \pi^{-1}\big(\overline{\pi(U)}\big)$$

are valid. Clearly $\overline{\pi(U)}$ is a regular closed set in H/N. Hence $\overline{\pi(U)}$ is closed in $(H/N)_{sr}$, where $(H/N)_{sr}$ is the semiregularization of the paratopological group H/N. From Theorem 2.7 it follows that $(H/N)_{sr}$ is a T_3-space. Hence Lemma 4.3 implies that $(H/N)_{sr}$ has a countable closed network as a continuous image of H/N. Thus $\overline{\pi(U)}$ is a G_δ-set in $(H/N)_{sr}$ and in H/N. Hence $\overline{U} = \pi^{-1}(\overline{\pi(U)})$ is a G_δ-set in H. This completes the proof. □

It was shown in [11, Proposition 2.6] that every regular $L\Sigma$-group G is *perfectly κ-normal*, i.e., the closure of every open subset of G is a zero-set.[4] Let us show that 'regular' can be weakened to 'Hausdorff' in this result. First we need a simple lemma.

Lemma 4.5. *Let $f : X \to Y$ be a continuous onto mapping, where X is Hausdorff and Y is regular. If X is a Lindelöf Σ-space, then so is Y.*

Proof. Let families $\mathcal{C}_X$ and $\mathcal{F}_X$ of closed subsets of X witness that X is a Lindelöf Σ-space. Then the family $\mathcal{C}_X$ is countable and $\mathcal{F}_X$ consists of compact sets. We claim that the families

$$\mathcal{C}_Y = \big\{\overline{f(C)}\colon\ C \in \mathcal{C}_X\big\} \quad \text{and} \quad \mathcal{F}_Y = \big\{f(F)\colon\ F \in \mathcal{F}_X\big\}$$

witness that Y is a Lindelöf Σ-space. Indeed, take an arbitrary element $F \in \mathcal{F}_X$ and let U be an open neighborhood of $f(F)$ in Y. Since $f(F)$ is compact and Y is regular, there exists an open neighborhood V of $f(F)$ in Y such that $\overline{V} \subset U$. Take an element $C \in \mathcal{C}$ such that $F \subset C \subset f^{-1}(V)$. Then $\overline{f(C)} \in \mathcal{C}_Y$ and $f(F) \subset f(C) \subset \overline{f(C)} \subset \overline{V} \subset U$. Since the family $\mathcal{C}_Y$ is countable, this proves our claim and the lemma. □

Proposition 4.6. *Every Hausdorff $L\Sigma$-group G is perfectly κ-normal.*

Proof. Let G_{sr} be the semiregularization of G and $i : G \to G_{sr}$ be the identity mapping. By Theorem 2.7, G_{sr} is a regular paratopological group. It follows from the definition of semiregularization that $i(\overline{O}) = \overline{i(O)}$, for every open set $O \subset G$. In other words, the families of regular closed sets in G and G_{sr} coincide. Lemma 4.5 implies that G_{sr} is a regular Lindelöf Σ-space, so G_{sr} is perfectly κ-normal by [11, Proposition 2.6]. Take a continuous real-valued function g on G_{sr} such that $\overline{i(O)} = g^{-1}(0)$. Then the continuous real-valued function $f = g \circ i$ on G satisfies $\overline{O} = f^{-1}(0)$. Hence G is perfectly κ-normal. □

Notice that the above result complements Proposition 4.4 in the case of Hausdorff paratopological groups.

For the proof of Theorem 4.8 we need the following auxiliary fact (see [11, Lemma 3.15]).

[4] A subset F of a space X is called a *zero-set* if there exists a continuous real-valued function f on X such that $F = f^{-1}(0)$.

Lemma 4.7. *Let D be a dense subspace of a space X, and suppose that $f : D \to Y$ and $\varphi : X \to Z$ are continuous mappings, where φ is open and the space Y is regular. Suppose also that $\mathcal{B}$ is a base for Y such that $\overline{f^{-1}(V)} = \varphi^{-1}(\varphi(\overline{f^{-1}(V)}))$, for each $V \in \mathcal{B}$ (the closures are taken in X). Then there exists a continuous mapping $g : \varphi(D) \to Y$ such that $f = g \circ \varphi\restriction_D$.*

The following theorem is one of the main results in this section. It answers Question 1.1 affirmatively and shows even more:

Theorem 4.8. *Every subgroup H of a totally $L\Sigma$-group G satisfying the T_1 separation axiom is $\mathbb{R}$-factorizable.*

Proof. By our assumption, the associated topological group G^* is a Lindelöf Σ-space. The closure of H in G^*, say, F is also a Lindelöf Σ-space. Then F, considered as a subgroup of G, is a paratopological group. We denote it by K. From (2) of Proposition 2.1 it follows that F is topologically isomorphic to the topological group K^* associated to K. Thus K is a totally $L\Sigma$-group. Clearly, H is dense in K. Therefore, we can assume without loss of generality that H is dense in G.

Let f be a continuous real-valued function on H. Denote by $\mathcal{B}$ a countable open base for the real line. Since f is continuous, for every $V \in \mathcal{B}$, there exists an open set U_V in G such that $f^{-1}(V) = U_V \cap H$. By Proposition 4.4, $\overline{U_V}$ is a G_δ-set in G, so we can apply (2) of Lemma 4.2 to find a closed invariant subgroup N of G such that the quotient paratopological group G/N has a countable network and the quotient homomorphism $\pi : G \to G/N$ satisfies $\overline{U_V} = \pi^{-1}(\pi(\overline{U_V}))$, for each $V \in \mathcal{B}$. Since H is dense in G, we have that $\overline{U_V} = \overline{f^{-1}(V)}$ for each $V \in \mathcal{B}$. Therefore, by Lemma 4.7 (with $D = H$, $X = G$, $Y = \mathbb{R}$, $Z = G/N$, and $\varphi = \pi$), there exists a continuous real-valued function g on $\pi(H)$ satisfying $f = g \circ \pi\restriction_H$.

Clearly, the subgroup $\pi(H)$ of G/N has a countable network. Hence $\pi(H)$ is $\mathbb{R}$-factorizable by Corollary 3.16. So we can find a continuous homomorphism $p : \pi(H) \to L$ onto a second-countable paratopological group L and a continuous real-valued function h on L such that $g = h \circ p$. Then $\varphi = p \circ \pi\restriction_H$ is a continuous homomorphism of H onto L and $f = h \circ \varphi$. This proves that H is $\mathbb{R}$-factorizable. □

Let H be a Hausdorff $L\Sigma$-group. According to [11, Corollary 2.3(b)], H^* is a Lindelöf Σ-group, i.e., every Hausdorff $L\Sigma$-group is a totally $L\Sigma$-group. Hence the following corollary to Theorem 4.8 extends Theorem 3.13 of [11] to Hausdorff paratopological groups.

Corollary 4.9. *Suppose that H is a Hausdorff $L\Sigma$-group. Then every subgroup of H is $\mathbb{R}$-factorizable.*

The next result generalizes [11, Proposition 3.16] by weakening 'regular' to 'T_1'.

Corollary 4.10. *Every subgroup of a σ-compact T_1 paratopological group is $\mathbb{R}$-factorizable.*

Proof. According to Theorem 4.8, it suffices to show that every σ-compact paratopological group G satisfying the T_1 separation axiom is totally σ-compact and, hence, a totally $L\Sigma$-group. For Hausdorff paratopological groups, this fact was established in [11, Corollary 2.3(a)]. We show here that a similar argument works in the case when 'Hausdorff' is weakened to 'T_1'.

Let τ be the topology of G. Denote by G' the paratopological group conjugated to G, i.e., $G' = (G, \tau^{-1})$, where $\tau^{-1} = \{U^{-1}\colon U \in \tau\}$. Then the inversion in G is a homeomorphism of G onto G'. Hence G' and the product $G \times G'$ are also σ-compact T_1 paratopological groups. By Lemma 2.2, the topological group G^* associated to G is topologically isomorphic to the diagonal $\Delta = \{(x, x) \in G \times G'\colon x \in G\}$ which is closed in $G \times G'$. This implies that $G^* \cong \Delta$ is a σ-compact Hausdorff topological group. So G is totally σ-compact. □

It is tempting to extend Corollary 4.10 to subgroups of paratopological groups satisfying the T_0 separation axiom:

Question 4.11. Is every subgroup of a σ-compact T_0 paratopological group $\mathbb{R}$-factorizable? What about dense subgroups?

We answer Question 4.11 affirmatively in the special case when the domain of a real-valued function is the whole group, without any separation restrictions on the group.

Corollary 4.12. *Every σ-compact paratopological group is $\mathbb{R}$-factorizable.*

Proof. Let G be a σ-compact paratopological group and f a continuous real-valued function on G. By Theorem 3.3, we can find a continuous homomorphism $\pi : G \to H$ onto a Hausdorff paratopological group H and a continuous real-valued function g on H such that $f = g \circ \pi$. Clearly, H is σ-compact. Then H is $\mathbb{R}$-factorizable according to Corollary 4.10, so there exist a continuous homomorphism $p : H \to K$ onto a second-countable paratopological group K and a continuous real-valued function h on K such that $g = h \circ p$. Hence $\varphi = p \circ \pi$ is a continuous homomorphism of G onto K which satisfies $f = h \circ \varphi$. This implies that G is $\mathbb{R}$-factorizable. □

Our next step is to extend Theorem 4.8 to dense subgroups of arbitrary products of totally $L\Sigma$-groups satisfying the T_1 separation axiom.

Corollary 4.13. *Let $G = \prod_{i \in I} G_i$ be the product of a family of T_1 paratopological groups. If each factor G_i is a totally $L\Sigma$-group, then every dense subgroup of G is $\mathbb{R}$-factorizable.*

Proof. Let S be a dense subgroup of G and f a continuous real-valued function on S. First we prove that $G = \prod_{i \in I} G_i$ is ω-cellular. By (4) of Proposition 2.1, we can identify the topological groups G^* and $\prod_{i \in I} G_i^*$. Since the topological group $\prod_{i \in F} G_i^*$ is a Lindelöf Σ-space (hence ω-cellular), for each finite set $F \subset I$, it follows from [2, Theorem 5.3.18] that $\prod_{i \in F} G_i^*$ is ω-cellular. Hence [2, Proposition 1.6.22] implies that the product space $G^* = \prod_{i \in I} G_i^*$ is ω-cellular, and so is G as a continuous image of G^*.

Since S is dense in G, it follows from [2, Theorem 1.7.7] that there exist a countable set $J \subset I$ and a continuous function $g : p_J(S) \to \mathbb{R}$ such that $f = g \circ p_J{\restriction}_S$, where $p_J : G \to \prod_{i \in J} G_i$ is the projection. It is well known that the product of a countable family of Lindelöf Σ-spaces is a Lindelöf Σ-space. Therefore, $\prod_{i \in J} G_i$ is a totally $L\Sigma$-group. Hence Theorem 4.8 implies that one can find a continuous homomorphism $\pi : p_J(S) \to H$ onto a second-countable paratopological group H and a continuous function $h : H \to \mathbb{R}$ such that $g = h \circ \pi$. Then $\varphi = \pi \circ p_J{\restriction}_S$ is a continuous homomorphism of S onto H, and $f = h \circ \varphi$. This completes the proof. □

The next fact is immediate from Corollary 4.13.

Corollary 4.14. *Let $G = \prod_{i \in I} G_i$ be the product of a family of Hausdorff paratopological $L\Sigma$-groups. Then every dense subgroup of G is $\mathbb{R}$-factorizable.*

The result below answers Question 1.2 affirmatively.

Corollary 4.15. *Let $G = \prod_{i \in I} G_i$ be the product of a family of T_1 paratopological groups. If each factor G_i is σ-compact, then every dense subgroup of G is $\mathbb{R}$-factorizable.*

5. Open problems

We formulate here several open problems whose solutions seem to require new methods, distinct from those employed in this article.

The first problem arises in an attempt to extend Proposition 3.10 to regular paratopological groups:

Question 5.1. Is every regular $\mathbb{R}$-factorizable paratopological group totally ω-narrow?

In view of Theorem 2.4, the above question is equivalent to asking whether every regular $\mathbb{R}$-factorizable paratopological group is Tychonoff. It is worth mentioning that there exist $\mathbb{R}$-factorizable paratopological groups satisfying the T_0 separation axiom which fail to be even ω-narrow.

Our second problem repeats the second part of [11, Problem 5.1].

Question 5.2. Let G be a (regular) paratopological group such that the associated topological group G^* is $\mathbb{R}$-factorizable. Is G then $\mathbb{R}$-factorizable?

We have already mentioned that if H is a Hausdorff $L\Sigma$-group, then the associated topological group H^* is a Lindelöf Σ-space. We do not know whether the converse is valid, unless H is regular:

Question 5.3. Suppose that H is a Hausdorff paratopological group such that the associated topological group H^* is a Lindelöf Σ-space. Is H a Lindelöf Σ-space?

Let H be a Hausdorff σ-compact paratopological group. By [10, Theorem 4.4], for every family γ of G_δ-sets in H, $\overline{\bigcup \gamma}$ is again a G_δ-set. It is not clear if this conclusion remains valid for Hausdorff $L\Sigma$-groups:

Question 5.4. Let H be a Hausdorff $L\Sigma$-group and γ a family of G_δ-sets in H. Is $\overline{\bigcup \gamma}$ a G_δ-set in H?

In fact, the above problem is equivalent to the following one (see [19, Problem 4.3]):

Question 5.5. Let H be a Hausdorff paratopological group with a countable network. Is every closed subset of H a G_δ-set?

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