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SOME PROPERTIES ON $\aleph_0$-WEAK BASES

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Abstract. In this paper, we characterize an $\aleph_0$-weakly first-countable space as a quotient, countable-to-one image of a first-countable space. We also discuss metrizability and mapping theorems on $\aleph_0$-weak bases, and pose some questions.

1. Introduction

Arhangel'skiĭ [1] introduced the concept of weak bases in 1966 and many interesting results related to weak bases, in particular, weakly first-countable spaces, have been obtained. As a generalization of weakly first-countability, Sirois-Dumais [35] introduced weakly quasi-first-countable spaces and proved that X is a weakly quasi-first-countable space if and only if X is a quotient, frontier-countable image of a metric space. Svetlichny [36] called a weakly quasi-first-countable space by an $\aleph_0$-weakly first-countable space and proved that there is a non-metrizable, $\aleph_0$-weakly first-countable topological group. In 99' Ohio University topology seminar, Arhangel'skiĭ called a weakly quasi-first-countable space by a σ-weakly first-countable space and posed some problems. The authors [21] introduced the concept of general $\aleph_0$-weak bases and proved a space X has a point-countable $\aleph_0$-weak base if and only if X is a quotient, countable-to-one image of a metric space. In this paper, we discuss some properties on $\aleph_0$-weak bases and pose some questions.

Throughout this paper, all spaces are assumed to be T_2, all maps are continuous and onto. Denote real, irrational and rational numbers by $\mathbb{R}, \mathbb{P}$ and $\mathbb{Q}$, respectively. We refer the reader to [7, 11] for notations and terminology not explicitly given here.

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2. Main Results

Definition 1. Let $\mathcal{B}$ be a family of subsets of a space X. $\mathcal{B}$ is said to be an $\aleph_0$-*weak base* for X if $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in \mathbb{N}\}$ satisfies

(1) For each $x \in X, n \in \mathbb{N}$, $\mathcal{B}_x(n)$ is closed under finite intersections and $x \in \cap\mathcal{B}_x(n)$.
(2) A subset U of X is open if and only if whenever $x \in U$ and $n \in \mathbb{N}$, there exists a $B_x(n) \in \mathcal{B}_x(n)$ such that $B_x(n) \subset U$.

For each $x \in X$, $\cup_{n\in\mathbb{N}}\mathcal{B}_x(n)$ is called an $\aleph_0$-weak base at x. X is called $\aleph_0$-*weakly first-countable* [36] or *weakly quasi-first-countable* in the sense of Sirois-Dumais [35] if $\mathcal{B}_x(n)$ is countable for each $x \in X$, $n \in \mathbb{N}$.

If $\mathcal{B}_x(n) = \mathcal{B}_x(1)$ for each $n \in \mathbb{N}$ in the definition of $\aleph_0$-weak bases, then $\mathcal{B}$ is said to be a *weak base* for X [1]. X is called *weakly first-countable* or *g-first-countable* in the sense of Arhangel'skiĭ [1] if $\mathcal{B}_x(1)$ is countable for each $x \in X$.

A space X is called a *sequential space* if each sequential open subset of X is open [7].

Lemma 2. [35] *Every $\aleph_0$-weakly first-countable space is sequential.*

The following lemma gives the relationship between weakly first-countability and $\aleph_0$-weakly first-countability. Call a subspace of a space X a *fan* (at a point $x \in X$) if it consists of a point x, and a countably infinite family of disjoint sequences converging to x. Call a subset of a fan a *diagonal* if it is a sequence meeting infinitely many of the sequences converging to x and converges to some point in the fan. A space X is an α_4-*space* [2] if every fan at x of X has a diagonal converging to x.

Lemma 3. [19] *A space X is weakly first-countable if and only if X is an $\aleph_0$-weakly first-countable, α_4-space.*

A space is strongly Fréchet-Urysohn (= strongly Fréchet = countably bisequential [28]) if whenever $x \in \cap_{n\in\mathbb{N}}\overline{A_n}$ with each $A_{n+1} \subset A_n$ then one may pick $x_n \in A_n$ such that $x_n \to x$. Every strongly Fréchet space is an α_4-space, and every Fréchet-Urysohn (= Fréchet), weakly first-countable space is first-countable [16]. By the lemma above, we have the following.

Corollary 4. *A space X is first-countable if and only if X is a strongly Fréchet-Urysohn, $\aleph_0$-weakly first-countable space.*

Proposition 5. *Suppose that each point of a space X is a G_δ-set and X is $\aleph_0$-weakly first-countable. Then X is weakly first-countable if and only if it contains no closed copy of the countable sequential fan S_ω.*

Proof. S_ω is not weakly first-countable. On the other hand, X is a sequential space in which every point is a G_δ-set. Then X is an α_4-space if it contains no closed copy of S_ω [30]. By Lemma 3, X is weakly first-countable. $\square$

Arhangel'skiĭ asked the following question: Is $\aleph_0$-weakly first-countable topological group metrizable? Svetlichny [36] gave a negative answer to the question, but we have the following.

For a Tychonoff space X, let α be a family of closed subsets of X. $C_\alpha(X)$ is the continuous real function space of X with the set-open topology [6, 27]. $C_\alpha(X)$ is always a topological group. There are two well-studied examples of set-open topologies, which is the space $C_p(X)$ with the point-open topology and the space $C_k(X)$ with the compact-open topology.

Corollary 6. *If the space $C_\alpha(X)$ with the set-open topology is $\aleph_0$-weakly first-countable, then it is metrizable.*

Proof. Since $C_\alpha(X)$ is $\aleph_0$-weakly first-countable, it is sequential by Lemma 2. Then it is strongly Fréchet-Urysohn [10]. By Corollary 4, $C_\alpha(X)$ is first-countable, hence metrizable [27]. $\square$

Question 7. Let X be $\aleph_0$-weakly first-countable and contain no closed copy of S_ω, is X weakly first-countable?

Question 8. (A. Arhangel'skiĭ) Let X be compact and $\aleph_0$-weakly first-countable, is X weakly first-countable?

Let $f : X \to Y$ be a map. f is called *subsequence-covering* [18] if whenever L is a convergent sequence in Y there is a compact subset K in X such that $f(K)$ is a subsequence of L.

Lemma 9. [17, 18] *Let $f : X \to Y$ be a map, and X a sequential space. Then f is quotient if and only if Y is a sequential space and f is subsequence-covering.*

Theorem 10. *The following are equivalent for a space X.*

(1) *X is an $\aleph_0$-weakly first-countable.*
(2) *X is a quotient, countable-to-one image of a first-countable space.*
(3) *X is a quotient, countable-to-one image of a weakly first-countable space.*

Proof. (1) $\Rightarrow$ (2). Assume X is $\aleph_0$-weakly first-countable, and let $\{C_x(n,m) : x \in X, n, m \in \mathbb{N}\}$ be an $\aleph_0$-weak base such that for $x \in X$, $C_x(n, m+1) \subset C_x(n,m)$ for each $n, m \in \mathbb{N}$. We rewrite $X = \{x_\alpha : \alpha \in \Gamma\}$, let $Y = \{y_\alpha : \alpha \in \Gamma\}$ and $X \cap Y = \emptyset$. Let $Z = X \cup Y$, fix $n \in \mathbb{N}$, we endow Z with a topology τ_n as follows: each point in X is open; for $y_\alpha \in Y$, the basic neighborhoods are $\{y_\alpha\} \cup (C_{x_\alpha}(n,m) \backslash \{x_\alpha\})$ $(m \in \mathbb{N})$. It is easy to see that (Z, τ_n) is first-countable.

Let $M = \bigoplus_{n\in\mathbb{N}}(Z, \tau_n)$. Then M is first-countable. Define $f : M \to X$ by $f(z) = x_\alpha$ if $z = x_\alpha$ or $z = y_\alpha$. It is not difficult to check that f is continuous, countable-to-one and onto. We prove that f is subsequence-covering. Let L be a sequence converging to $x_\alpha \in X$, then there are $n \in \mathbb{N}$ and a subsequence $L_1 \subset L$ such that L_1 is eventually in each $C_{x_\alpha}(n, m)$ for $m \in \mathbb{N}$ [34]. By the definition of τ_n, L_1 is still a sequence converging to $y_\alpha \in Z$ with $f(L_1) = L_1$ and $f(y_\alpha) = x_\alpha$. Hence f is a subsequence-covering map. Since X is a sequential space, f is a quotient map by Lemma 9.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial. □

Question 11. Let X be an $\aleph_0$-weakly first-countable space. Is X a quotient, countable-to-one image of a regular first-countable space?

Since the composition of two quotient, countable-to-one maps is still a quotient, countable-to-one map, we have the following.

Corollary 12. *The $\aleph_0$-weakly first-countability is preserved by quotient, countable-to-one maps.*

Lemma 13. [20] *Let $f : X \to Y$ be a closed map from a normal Fréchet-Urysohn space X to a space Y. If Y is $\aleph_0$-weakly first-countable, then the boundary $\partial f^{-1}(y)$ is σ-compact for each $y \in Y$. In particular, the ω_1-fan S_{ω_1} is not $\aleph_0$-weakly first-countable.*

The $\aleph_0$-weakly first-countability may not be preserved by perfect mappings. In fact, Gruenhage, Michael and Tanaka [12, Example 9.8] constructed a space Y that is a perfect image of a weakly first-countable space. Since Y contains a closed copy of S_{ω_1}, Y is not $\aleph_0$-weakly first-countable by Lemma 13.

Arhagel'skiĭ asked the following question in 99' Ohio University topology seminar.

Question 14. Let X be a closed image of a separable metric space with $|X| \leq \omega$, is X $\aleph_0$-weakly first-countable?

We shall give a negative answer to the above question.

Example 15. There is a countable, closed image of a separable metric space that is not $\aleph_0$-weakly first-countable.

Proof. Let $M = (\mathbb{Q}\times(\mathbb{Q}\backslash\{0\}))\cup(\mathbb{P}\times\{0\})$, endow M with usual topology, and let X be the quotient space by identifying $(\mathbb{P} \times \{0\})$ to be a point ∞. The quotient map from M onto X is a closed map. Then X is a closed image of a separable metric space with $|X| \leq \omega$. If X is $\aleph_0$-weakly first-countable, the boundary of $f^{-1}(\infty)$ is σ-compact by Lemma 13, this is a contradiction since $\mathbb{P}$ is not σ-compact. □

Recall some basic concepts. Let $\mathcal{P}$ be a family of subsets of a space X. Then $\mathcal{P}$ is called a *k-network* [11] for X if for any compact set K and for any open set U with $K \subset U$, $K \subset \cup\mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is called a *cs^*-network* [9] for X if for any sequence L converging to $x \in X$ and for any open set U with $x \in U$, there exist a subsequence L' of L and a $P \in \mathcal{P}$ such that $L' \cup \{x\} \subset P \subset U$. A space is an $\aleph_0$-space (resp. $\aleph$-space) if it is a regular space with a countable (resp. σ-locally finite) k-network.

Lemma 16. [34] *Every $\aleph_0$-weak base for a space is a cs^*-network.*

It was proved [24] that a cosmic space (= a regular space with a countable network) with a point-countable weak base has a countable weak base. But, a cosmic space with a point-countable $\aleph_0$-weak base need not have a countable $\aleph_0$-weak base. Since every regular space with a countable cs^*-network is an $\aleph_0$-space [9], every regular space with a countable $\aleph_0$-weak base is an $\aleph_0$-space by Lemma 16. Under the assumption that there exists a σ'-set, Sakai [32] constructed a cosmic space X that is a quotient, finite-to-one image of a metric space, which is not an $\aleph_0$-space. Since a quotient, finite-to-one image of a metric space has a point-countable $\aleph_0$-weak base [21], the space X is not an $\aleph_0$-space, hence it has no countable $\aleph_0$-weak base.

Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is called *closure-preserving* if $\overline{\cup\mathcal{P}'} = \cup\{\overline{P} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$. $\mathcal{P}$ is called *hereditarily closure-preserving* (abbr. *HCP*) if every family $\{H(P) : P \in \mathcal{P}\}$ with $H(P) \subset P \in \mathcal{P}$ is closure-preserving. $\mathcal{P}$ is called *point-discrete* (i.e., *weakly hereditarily closure-preserving* in sense of Burke, Engelking and Lutzer [4]) if each set $\{x(P) : P \in \mathcal{P}\}$ with $x(P) \in P \in \mathcal{P}$ is a closed discrete subset of X. $\mathcal{P}$ is called *compact-finite* if every compact set of X intersects only finitely many members of the family $\mathcal{P}$.

Lemma 17. *Let $\mathcal{P}$ be a point-discrete family of a space X.*

(1) *If there is a non-trivial convergent sequence L such that L is eventually in each element in $\mathcal{P}$, then $\mathcal{P}$ is finite.*
(2) *If X is Fréchet-Urysohn, then $\mathcal{P}$ is HCP* [15].

Proof. We show that the (1) holds. If $\mathcal{P}$ is not finite, there exists an infinite subset $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}$. Since L is eventually in each P_n, there is a subsequence $\{x_n\}_{n\in\mathbb{N}} \subset L$ such that $x_n \in P_n$ for each $n \in \mathbb{N}$. Then the set $\{x_n : n \in \mathbb{N}\}$ is closed discrete in X, a contradiction. $\square$

Theorem 18. *The following are equivalent for a regular space X.*

(1) *X has a σ-discrete $\aleph_0$-weak base.*
(2) *X has a σ-locally finite $\aleph_0$-weak base.*

(3) *X has a σ-HCP $\aleph_0$-weak base.*
(4) *X is an $\aleph_0$-weakly first-countable space with a σ-HCP k-network.*
(5) *X has a σ-compact-finite $\aleph_0$-weak base consisting of closed subsets.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (2) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (2). Let X have a σ-compact-finite $\aleph_0$-weak base consisting of closed subsets. Then X is a sequential space, thus it is a k-space. Every compact-finite family of closed subsets of X is closure-preserving because X is a k-space. Since every closure-preserving and point-finite family of closed subsets is locally finite, every compact-finite family of closed subsets in X is locally finite. Hence, X has a σ-locally finite $\aleph_0$-weak base.

(3) $\Rightarrow$ (4). Let $\mathcal{B}$ be a σ-HCP $\aleph_0$-weak base for X. Then $\mathcal{B}$ is a σ-HCP cs^*-network by Lemma 16, thus it is a σ-HCP k-network [9]. Next, we shall prove that X is $\aleph_0$-weakly first-countable. Put

$$\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in \mathbb{N}\} = \cup_{m\in\mathbb{N}}\mathcal{B}_m,$$

where each $\mathcal{B}_x(n)$ is of the properties in Definition 1 and each $\mathcal{B}_m$ is HCP. Assume that each element in $\mathcal{B}$ is closed in X by the regularity of X. Fix $n \in \mathbb{N}$, a non-isolated point $x \in X$, we prove that $\mathcal{B}_x(n)$ is countable. Since $\mathcal{B}_x(n) = \cup_{m\in\mathbb{N}}(\mathcal{B}_m \cap \mathcal{B}_x(n))$, and $\mathcal{B}_m \cap \mathcal{B}_x(n)$ is HCP for each $m \in \mathbb{N}$, so we only need to prove that there is a non-trivial converging sequence L such that L is eventually in each $B \in \mathcal{B}_x(n)$ by Lemma 17.

Since X has a σ-HCP k-network, it is a σ-space, thus each point of X is a G_δ-set [11]. There is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open subsets of X such that $\{x\} = \cap_{i\in\mathbb{N}}U_i$ with each $\overline{U}_{i+1} \subset U_i$. Pick $x(B, m) \in U_m \cap B - \{x\}$ for each $m \in \mathbb{N}$ and $B \in \mathcal{B}_m \cap \mathcal{B}_x(n)$. Let

$$M = \{x\} \cup \{x(B, m) : m \in \mathbb{N}, B \in \mathcal{B}_m \cap \mathcal{B}_x(n)\}.$$

Then M is a closed subspace of X and x is the only non-isolated point in M. It is not difficult to see that M has a σ-HCP $\aleph_0$-weak base $\{B \cap M : B \in \mathcal{B}\}$. We endow M with a new topology as follows: each point in M except x is open, the neighborhood base of x is $\{B \cap M : B \in \mathcal{B}_x(n)\}$. We denote the new space by M'. By the definition, M' is regular and the topology on M' is finer than the topology on M. So $\{B \cap M : B \in \mathcal{B}_x(n)\}$ is σ-HCP in M', hence M' has a σ-HCP base. Then M' is metrizable [4]. Therefore, there is a non-trivial sequence L converging to x in M'. Since $\{B \cap M : B \in \mathcal{B}_x(n)\}$ is a local base at x in M', L is eventually in each element in $\{B \cap M : B \in \mathcal{B}_x(n)\}$. Hence L is eventually in each $B \in \mathcal{B}_x(n)$. Thus X is $\aleph_0$-weakly first-countable.

(4) $\Rightarrow$ (1). By Lemma 13, X contains no closed copy of S_{ω_1}. Then X is an $\aleph$-space since a regular space having a σ-HCP k-network is an $\aleph$-space if it contains no closed copy of S_{ω_1} [14]. It is not difficult to prove that an $\aleph_0$-weakly first-countable, $\aleph$-space has a σ-discrete $\aleph_0$-weak base [34]. $\square$

Since σ-hereditarily closure-preserving k-networks are preserved by closed maps in regular spaces [17], and $\aleph_0$-weakly first-countability is preserved by quotient, countable-to-one maps, we have the following.

Proposition 19. *In regular spaces, spaces with a σ-locally finite $\aleph_0$-weak base are preserved by closed, countable-to-one maps.*

It is not difficult to prove that closed and open maps preserve σ-locally-finite $\aleph_0$-weak base, but the authors do not know if perfect maps preserve σ-locally-finite $\aleph_0$-weak base. A map $f : X \to Y$ is called *σ-compact* if each $f^{-1}(y)$ is a σ-compact subset in X for each $y \in Y$.

Theorem 20. *The following are equivalent for a regular space X.*

(1) *X is an image of a locally separable metric space under a closed and σ-compact map.*
(2) *X is a Fréchet-Urysohn space with a star-countable $\aleph_0$-weak base.*
(3) *X is a topological sum of Fréchet-Urysohn spaces each with a countable $\aleph_0$-weak base.*
(4) *X is a Fréchet-Urysohn space with a point-countable $\aleph_0$-weak base consisting of separable subsets.*

Proof. (1) $\Rightarrow$ (3). Let X be an image of a locally separable metric space under a closed and σ-compact map. Then X is a Fréchet-Urysohn space with a star-countable k-network by [23, Theorem 1.9], and X is an $\aleph_0$-weakly first-countable, $\aleph$-space by [20, Theorem 2.2]. Thus X is a topological sum of Fréchet-Urysohn and $\aleph_0$-spaces by [31, Theorem 2.6]. Since every $\aleph_0$-weakly first-countable, $\aleph_0$-space has a countable $\aleph_0$-weak base [21], X is a topological sum of Fréchet-Urysohn spaces each with a countable $\aleph_0$-weak base.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Every point-countable cs^*-network is a k-network for a sequential space [37]. Then X is a Fréchet-Urysohn space with a star-countable k-network by Lemma 16, so X is the image of a locally separable metric space under a closed map f by [23, Theorem 1.9]. By Lemma 13, the boundary $\partial f^{-1}(y)$ is σ-compact for each $y \in Y$. Then X is the image of a locally separable metric space under a closed and σ-compact map by the similar proof in [7, Theorem 4.4.17].

(3) $\Rightarrow$ (4). Since every space with a countable $\aleph_0$-weak base is hereditarily separable, it is trivial.

(4) $\Rightarrow$ (3). Let $\mathcal{P}$ be a point-countable $\aleph_0$-weak base consisting of separable subsets. For $x \in X$, by [12, Lemma 2.6], $x \in \mathrm{int}(\mathrm{st}(x, \mathcal{P}))$, hence X is locally separable. By [12, Proposition 8.8], X is a topological sum of $\aleph_0$-subspaces since every point-countable $\aleph_0$-weak base is a point-countable k-network by the proof of (2) $\Rightarrow$ (1). By [21, Theorem 7], X is a topological sum of Fréchet-Urysohn spaces each with a countable $\aleph_0$-weak base. $\square$

In [21], we proved that X is an image of a separable metric space under a quotient σ-compact map if and only if X is an image of a separable metric space under a quotient countable-to-one map. But the following is still unknown.

Question 21. Is an image of a separable metric space under a closed σ-compact map an image of a separable metric space under a closed countable-to-one map?

Definition 22. A space X is called a *κ-Fréchet-Urysohn* space [22] if whenever $x \in \overline{U}$ with U open, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset U$ such that $x_n \to x$.

A space X is *strongly κ-Fréchet-Urysohn* [33] at a point $x \in X$ if for each sequence $\{O_n\}_{n\in\mathbb{N}}$ of decreasing open subsets with $x \in \cap_{n\in\mathbb{N}}\overline{O_n}$, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x with each $x_n \in O_n$. A space X is *strongly κ-Fréchet-Urysohn* if it is strongly κ-Fréchet-Urysohn at each point of X.

Every Fréchet-Urysohn space is κ-Fréchet-Urysohn, but need not be strongly κ-Fréchet-Urysohn, for example S_ω. It is easy to see that strongly κ-Fréchet-Urysohn is κ-Fréchet-Urysohn. Sakai [33] proved that every κ-Fréchet-Urysohn topological group is strongly κ-Fréchet-Urysohn. In [22], there exists a κ-Fréchet-Urysohn topological group that is not a k-space or has no countable tightness. Hence, a strongly κ-Fréchet-Urysohn space need not be a k-space or have countable tightness.

Lemma 23. [33] *Let X be a space and $x \in X$. If X has a countable cs^*-network at x and is strongly κ-Fréchet-Urysohn at x, then the point x has a countable neighborhood base.*

Theorem 24. *A regular space X is metrizable if and only if X is a strongly κ-Fréchet-Urysohn space with a σ-point-discrete $\aleph_0$-weak base.*

Proof. Necessity is trivial.

Sufficiency. We first prove that X is a first-countable space. Let $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ be a σ-point-discrete $\aleph_0$-weak base, where each $\mathcal{B}_n$ is point-discrete. Fix $x \in X$, if x is an isolated point, then X is first-countable at x; otherwise, we shall prove that X has a countable cs^*-network at x, then X has a countable base at x by Lemma 23. Let $\mathcal{B}_x = \cup_{n\in\mathbb{N}}\mathcal{B}_x(n) \subset \mathcal{B}$ be the $\aleph_0$-weak base at the non-isolated point x. Put

$$\mathbb{N}_1 = \{k \in \mathbb{N} : \text{there is a non-trivial convergent sequence } L_1 \\ \text{that is eventually in each } B \in \mathcal{B}_x(k)\}.$$

Then $\{\mathcal{B}_x(k) : k \in \mathbb{N}_1\}$ is a cs^*-network at x. In fact, let L be a sequence converging to $x \in U$ with U open. We can assume that L is non-trivial, otherwise, since $x \in \overline{X - \{x\}}$, there exists a non-trivial sequence S converging to x in X by the κ-Fréchet-Urysohn property, then $L \cup S$ is a non-trivial sequence converging to x. There exist an $n_0 \in \mathbb{N}$ and a subsequence $L_1 \subset L$ such that L_1 is eventually in each $B \in \mathcal{B}_x(n_0)$ [34], then $n_0 \in \mathbb{N}_1$. By Definition 1, there exists $B \in \mathcal{B}_x(n_0)$ with $B \subset U$, and L_1 is eventually in B. Thus $\{\mathcal{B}_x(k) : k \in \mathbb{N}_1\}$ is a cs^*-network at x.

For each $n \in \mathbb{N}$ and $k \in \mathbb{N}_1$, the family $\mathcal{B}_n \cap \mathcal{B}_x(k)$ is point-discrete, thus it is finite by Lemma 17. Hence $\mathcal{B}_x(k) = \cup_{n\in\mathbb{N}}(\mathcal{B}_n \cap \mathcal{B}_x(k))$ is countable for each $k \in \mathbb{N}_1$. Therefore X has a countable cs^*-network at x, thus X is first-countable at x. At this stage, we have proved that X is first-countable.

X has a σ-HCP cs^*-network by Lemmas 16 and 17. Thus X has a σ-HCP k-network [9]. Since X is first-countable, it is metrizable [8]. □

Note 25. We can not replace "strongly κ-Fréchet-Urysohn" with "Fréchet-Urysohn" in Theorem 24. The sequential fan S_ω is a non-metrizable, Fréchet-Urysohn space with a countable $\aleph_0$-weak base.

Proposition 26. *If $C_p(X)$ has a σ-point-discrete $\aleph_0$-weak base, then X is countable.*

Proof. In view of the proof of Theorem 24, $\{\mathcal{B}_{\mathbf{0}}(k) : k \in \mathbb{N}_1\}$ is a countable cs^*-network at $\mathbf{0}$ in $C_p(X)$. Sakai [33] proved that if $C_p(X)$ has a countable cs^*-network at $\mathbf{0}$, then X is countable. Hence X is countable. □

Note 27. A space with a σ-point-discrete $\aleph_0$-weak base need not be a k-space or have countable tightness, in fact, Burke, Engelking and Lutzer [4] constructed a space with a σ-point-discrete base, which is neither k-space nor of countable tightness.

Question 28. Suppose that $C_k(X)$ have a σ-point-discrete $\aleph_0$-weak base, is $C_k(X)$ metrizable?

Let $\mathcal{P}$ be a cover of a space X. $\mathcal{P}$ is *cs-regular* [13] if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x and each open neighborhood U of x, there exists $m \in \mathbb{N}$ such that $\{P \in \mathcal{P} : P \cap T(x,m) \neq \emptyset, P \not\subset U\}$ is finite, where $T(x,m) = \{x\} \cup \{x_n : n > m\}$. Jiang [13] proved that a space is metrizable if and only if it has a cs-regular base. We sharpen his theorem by giving the following.

Theorem 29. *A space X is metrizable if and only if X has a cs-regular $\aleph_0$-weak base.*

Proof. In [38, Theorem 2.2] P. Yan and S. Lin proved that a space X is metrizable if and only if X is a sequential space with a cs-regular cs^*-network. To complete the proof, we only need to show that X is $\aleph_0$-weakly first-countable by Lemma 16.

Let $\mathcal{P} = \cup\{\mathcal{P}_x(n) : x \in X, n \in \mathbb{N}\}$ is a cs-regular $\aleph_0$-weak base of X. For each $x \in X$, put $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. Suppose $(\mathcal{P})_x$ is uncountable for some $x \in X$. Since $\mathcal{P}$ is cs-regular, it is easy to see the following:

(i) for each $y \neq x$, $\{P \in (\mathcal{P})_x : y \in P\}$ is finite.

(ii) for each infinite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$, $\mathcal{P}'$ forms a network at x, that is, $P \subset U$ for some $P \in \mathcal{P}'$ whenever $x \in U \in \tau$.

For each $P \in (\mathcal{P})_x$, pick $y(P) \in P - \{x\}$, then $(\mathcal{P})_x \cap (\mathcal{P})_{y(P)}$ is finite by (i). Since

$$(\mathcal{P})_x = \bigcup_{n\in\mathbb{N}} \{P \in (\mathcal{P})_x : |(\mathcal{P})_x \cap (\mathcal{P})_{y(P)}| = n\}$$

is uncountable, there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{P}_0 = \{P \in (\mathcal{P})_x : |(\mathcal{P})_x \cap (\mathcal{P})_{y(P)}| = k_0\}$$

is uncountable. By (i), $\{y(P) : P \in \mathcal{P}_0\}$ is uncountable. So we can choose $\{P_n : n \in \mathbb{N}\} \subset (\mathcal{P})_x$ and $\{x_n\}_{n\in\mathbb{N}}$ satisfying that $x_n \in P_n - \{x\}$ and $|(\mathcal{P})_x \cap (\mathcal{P})_{x_n}| = k_0$ for each $n \in \mathbb{N}$, where $P_n \neq P_{n'}$ and $x_n \neq x_{n'}$ for $n \neq n'$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges to x by (ii). Since $\mathcal{P}$ is a cs^*-network of X, there exist $\{Q_i : i \in \mathbb{N}\} \subset (\mathcal{P})_x$ and $\{n_i : i \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\{x_{n_j} : j \geq i\} \subset Q_i \subset X - \{x_{n_j} : j < i\}$$

for each $i \in \mathbb{N}$ by the induction. Pick $i_0 > k_0$, then $|(\mathcal{P})_x \cap (\mathcal{P})_{x_{n_{i_0}}}| \geq i_0 > k_0$, a contradiction. Therefore $(\mathcal{P})_x$ is countable. Thus X is $\aleph_0$-weakly first-countable. $\square$

Note 30. Martin [26] proved the following result. A space X is metrizable if and only if there exists a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of covers of X such that for each $x \in X$, $\{\mathrm{st}^2(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a weak base at x. We can not replace "weak base" with "$\aleph_0$-weak base" in the above result. For example, let $X = \{\infty\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ be a copy of S_ω, where the sequence $\{x_n(m)\}_{n\in\mathbb{N}}$ converges to ∞ for each $m \in \mathbb{N}$. For each $m, n \in \mathbb{N}$, let

$$L_{m,n} = \{\infty\} \cup \{x_i(m) : i \geq n\}, \text{and}$$

$$\mathcal{L}_{m,n} = \{\{x\} : x \in X \backslash \{\infty\}\} \cup \{L_{m,n}\}.$$

Define a sequence $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$ as follows:

$$\mathcal{G}_1 = \mathcal{L}_{1,1},\ \mathcal{G}_2 = \mathcal{L}_{1,2},\ \mathcal{G}_3 = \mathcal{L}_{2,2},\ \mathcal{G}_4 = \mathcal{L}_{1,3},\ \mathcal{G}_5 = \mathcal{L}_{2,3},\ \cdots\cdots .$$

This is that $\mathcal{G}_k = \mathcal{L}_{m,n}$, where $k = m + \frac{n(n-1)}{2}, m \leq n$. For each $x \in X$, it is straightforward to prove that $\{\mathrm{st}^2(x, \mathcal{G}_k) : k \in \mathbb{N}\}$ is an $\aleph_0$-weak base at x.

Let X be a regular space. X is called an M_1-space if it has a σ-closure-preserving base, an M_2-space if it has a σ-closure-preserving quasi-base, an M_3-space if it has a σ-cushioned pair-base [5]. Obviously, M_1-spaces $\Rightarrow M_2$-spaces $\Rightarrow M_3$-spaces [5]. H. J. K. Junnila, G. Gruenhage proved that every M_3-space is M_2, see [11]. It is still an open problem whether every M_3-space is M_1 [5]. The proof of the following theorem is based on Foged's proof on $\aleph$-spaces [11].

Lemma 31. [29, Theorem 15] *Let X be an M_3-space with property (P):*

Whenever U is open and $x \in \overline{U}\backslash U$, there exists a closure-preserving collection $\mathcal{F}$ of closed subsets of X that is a network at x and $\overline{F \cap U} = F$ for each $F \in \mathcal{F}$.

Then X is an M_1-space such that every closed subset has a closure-preserving open neighborhood base in X.

Theorem 32. *Let X be strongly κ-Fréchet-Urysohn. Then X is an M_1-space if and only if it is a regular space with a σ-closure-preserving $\aleph_0$-weak base.*

Proof. We only need to prove sufficiency. Let $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ be a σ-closure-preserving $\aleph_0$-weak base for X, where $\mathcal{B}_n$ is closure-preserving for each $n \in \mathbb{N}$. Since X is regular, we may assume that each element of $\mathcal{B}$ is closed in X. Since every $\aleph_0$-weak base is a network, X is a σ-space, therefore semi-stratifiable [11]. We first prove that strongly Fréchet-Urysohn spaces with a σ-closure-preserving $\aleph_0$-weak base are monotonically normal.

Let H and K be disjoint closed subsets of X. For each $n \in \mathbb{N}$, let

$$U_n = \cup\{F \in \mathcal{B}_i : i \leq n, F \cap K = \emptyset\} \backslash \cup \{F \in \mathcal{B}_i : i \leq n, F \cap H = \emptyset\}$$

and let

$$D(H, K) = \mathrm{int}(\cup_{n\in\mathbb{N}}\overline{U_n}).$$

It is easy to see that D is "monotone" with respect to each pair (H, K) of disjoint closed subsets of X. We prove that $H \subset D(H, K)$ and $\overline{D(H, K)} \subset X\backslash K$.

If there exists $x \in H\backslash D(H, K)$, then $x \in \overline{X\backslash \cup_{n\in\mathbb{N}} \overline{U_n}}$. For each $n \in \mathbb{N}$, $x \in \overline{X\backslash \cup_{i\leq n} \overline{U_i}}$. Since X is strongly κ-Fréchet-Urysohn, there is $x_n \in X\backslash \cup_{i\leq n} \overline{U_i}$ with $x_n \to x$. Since $\mathcal{B}$ is an $\aleph_0$-weak base, some $F \in \mathcal{B}_m$ contains infinitely many $x_n's$ and does not meet K by Lemma 16.

Since $x \in X \backslash \cup \{F \in \mathcal{B}_m : F \cap H = \emptyset\}$, we can pick $j > m$ with $x_j \in U_m$. On the other hand, $x_j \in X \backslash \cup_{i \leq j} \overline{U_i}$, this is a contradiction. Hence $H \subset D(H, K)$.

We prove that $\overline{D(H,K)} \subset X \backslash K$. Suppose that $x \in K \cap \overline{D(H,K)}$, there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset D(H,K)$ with $x_n \to x$ since X is κ-Fréchet-Urysohn. Some $F \in \mathcal{B}_m$ contains a subsequence L of $\{x_n\}$ and $F \cap H = \emptyset$. By the construction of U_n, $U_i \cap F = \emptyset$ for $i \geq m$. Thus

$$L \subset \cup_{i \leq m} \overline{U_i} \subset \cup \{F \in \mathcal{B}_i : i \leq m, F \cap K = \emptyset\}.$$

Then there exists i_0 such that

$$x \in \overline{U_{i_0}} \subset \overline{\cup \{F \in \mathcal{B}_i : i \leq i_0, F \cap K = \emptyset\}} = \cup \{F \in \mathcal{B}_i : i \leq i_0, F \cap K = \emptyset\},$$

hence $x \notin K$. This is a contradiction.

Therefore X is monotonically normal, thus M_3 [11]. Let U be an open subset of X, and let $x \in \overline{U} \backslash U$. Since X is κ-Fréchet-Urysohn, there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset U$ such that $x_n \to x$. Let $\mathcal{F} = \{L_i : i \in \mathbb{N}\}$, where each $L_i = \{x_n : n \geq i\} \cup \{x\}$. It is easy to check that $\mathcal{F}$ is a closure-preserving family consisting of closed subsets of X and a network at x, also $\overline{L_i \cap U} = L_i$. By Lemma 31, X is an M_1-space. $\square$

Question 33. Let X be a κ-Fréchet-Urysohn space. If X is a regular space with a σ-closure-preserving $\aleph_0$-weak base, is X an M_1-space?

Question 34. Is there a normal, $\aleph_0$-weakly first-countable space which is not collectionwise Hausdorff in ZFC?

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