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THE CLOSED MAPPINGS ON k-SEMISTRATIFIABLE SPACES

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Abstract. Let $f : X \to Y$ be a closed mapping, where X is a k-semistratifiable k-space. If Y contains no closed copy of S_{ω_1}(resp. S_ω), then $\partial f^{-1}(y)$ is Lindelöf(resp. compact) for each $y \in Y$. This improves some results about closed mappings on generalized metric spaces obtained by Liu [10], Tanaka [13, 14, 15], Tanaka and Liu [16], and Yun [19]. At last, two mapping theorems on $k\beta^+$-spaces are established.

1. Introduction

The following Hanai-Morita-Stone Theorem(see [3]) is well known. Let $f : X \to Y$ be a closed mapping, where X is a metric space. Then $\partial f^{-1}(y)$ is compact for each $y \in Y$ if and only if Y is a metric space.

Y. Tanaka [13, 14, 15] proved the following theorem.

Theorem 1.1. *Let $f : X \to Y$ be a closed mapping, where X is a normal, k-and $\aleph$-space. Then $\partial f^{-1}(y)$ is Lindelöf(resp. compact) for each $y \in Y$ if and only if Y contains no closed copy of S_{ω_1}(resp. S_ω).*

And the following question was posed by Y. Tanaka and Chuan Liu [16].

Question 1.2. *Let $f : X \to Y$ be a closed map. Under what conditions on X or Y, does $\partial f^{-1}(y)$ have some nice properties for each $y \in Y$?*

Interestingly, Liu [10] and Yun [19] have obtained a more precise result recently.

Theorem 1.3. *Let $f : X \to Y$ be a closed mapping, where X is a k-and-$\aleph$-space. Then $\partial f^{-1}(y)$ is Lindelöf(resp. compact) for each $y \in Y$ if and only if Y contains no closed copy of S_{ω_1}(resp. S_ω).*

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Definition 1.4. [11] A space X is said to be k-semistratifiable if for each open subset U of X there is a sequence $\{F(n,U)\}_{n\in\mathbb{N}}$ of closed subsets of X such that

(1) $U=\bigcup_{n\in\mathbb{N}} F(n,U)$;
(2) If $V\subset U$, then $F(n,V)\subset F(n,U)$;
(3) If a compact subset $K\subset U$, then $K\subset F(m,U)$ for some $m\in\mathbb{N}$.

Each $\aleph$-space is a k-semistratifiable space [11]. In the paper, we point out that the sufficiency still holds if X in Theorem 1.3 is weakened to be a k-semistratifiable k-space(see Theorem 2.5), and it is not true if X is a Moore space(see Remark 2.6) or an $\aleph$-space(see Remark 2.7).

On the other hand, k-semistratifiable spaces are preserved by closed mapping [5]. Each k-semistratifiable space is a $k\beta$-space, and each $k\beta$-space is preserved by compact-covering and closed mappings [17], here a continuous mapping $f: X\to Y$ is called a compact-covering mapping [3] if K is compact in Y, then $f(L)=K$ for some compact subset L in X. The following question is still open [17, Question 3.5].

Question 1.5. *Is each $k\beta$-space preserved by closed mappings?*

In this paper $k\beta^+$-spaces are introduced and discussed, and it is proved that $k\beta^+$-spaces are preserved by closed mappings(see Theorem 3.3).

All spaces are assumed to be Hausdorff, and mappings are continuous and surjective.

2. Main Results

Let X be a space and $P\subset X$. P is said to be a sequential neighborhood of $x\in P$ in X if each sequence converging to x is eventually in P. P is a sequentially open subset of X if P is a sequential neighborhood of x in X for each $x\in P$. P is a sequentially closed subset of X if $X\setminus P$ is sequentially open. X is said to be a sequential space [3] if each sequentially open subset is open in X.

Lemma 2.1. [8] *Let X be a k-semistratifiable space. Then for each subset W of X there is a sequence $\{H(n,W)\}_{n\in\mathbb{N}}$ of closed subsets of X such that*

(1) *$H(n,W)\subset H(n+1,W)\subset W$;*
(2) *If $V\subset W$, then $H(n,V)\subset H(n,W)$;*
(3) *If W is a sequential neighborhood of x, then every sequence converging to x is eventually in $H(m,W)$ for some $m\in\mathbb{N}$;*
(4) *If $\{G_\alpha:\alpha\in\Lambda\}$ is a disjoint family of subsets of X and $n\in\mathbb{N}$, then $\{H(n,G_\alpha):\alpha\in\Lambda\}$ is a discrete family in X.*

A subset D of a space X is said to be relatively discrete in X if D is a discrete subspace of X, i.e., for each $x \in D$, there is an open neighborhood U_x of x such that $U_x \cap (D \setminus \{x\}) = \emptyset$.

Lemma 2.2. *Let X be a k-semistratifiable space. If $D = \{x_\alpha : \alpha \in \Lambda\}$ is a relatively discrete subset of X, there is a disjoint family $\{U_\alpha : \alpha \in \Lambda\}$ such that*

(1) *U_α is a sequential neighborhood of x_α in X for each $\alpha \in \Lambda$;*
(2) *$\{y_\alpha : \alpha \in \Lambda'\} \cup \overline{D}$ is sequentially closed in X for each $\Lambda' \subset \Lambda$ and $y_\alpha \in U_\alpha$.*

Proof. Suppose that $H(\cdot, \cdot)$ is a function, which satisfies Lemma 2.1. For $\alpha \in \Lambda$, let

(1) $L_\alpha = \overline{\{x_\beta : \alpha \neq \beta \in \Lambda\}}$;
(2) $G_\alpha = \bigcup_{n \in \mathbb{N}} (H(n, X \setminus L_\alpha) \setminus H(n, X \setminus \{x_\alpha\}))$; and
(3) $U_\alpha = \bigcup_{n \in \mathbb{N}} (H(n, G_\alpha) \setminus H(n, X \setminus \{x_\alpha\}))$.

Then U_α is a sequential neighborhood of x_α in X. In fact, suppose a sequence $S \to x_\alpha$. Since $x_\alpha \notin L_\alpha$, by Lemma 2.1(3), S is eventually in some $H(m, X \setminus L_\alpha)$. Thus S is eventually in $H(m, X \setminus L_\alpha) \setminus H(m, X \setminus \{x_\alpha\}) \subset G_\alpha$. Hence S is eventually in some $H(k, G_\alpha) \setminus H(k, X \setminus \{x_\alpha\}) \subset U_\alpha$.

It is easy to check that $\{G_\alpha : \alpha \in \Lambda\}$ is disjoint and $U_\alpha \subset G_\alpha$. Then $\{U_\alpha : \alpha \in \Lambda\}$ is disjoint. If there is $\Lambda' \subset \Lambda$ such that $\{y_\alpha : \alpha \in \Lambda'\} \cup \overline{D}$ is not sequentially closed in X with some $y_\alpha \in U_\alpha$ for each $\alpha \in \Lambda'$, then there is a non-trivial sequence L in $\{y_\alpha : \alpha \in \Lambda'\} \setminus \overline{D}$ such that L converges to some point $x \notin \overline{D}$. We can assume that there is an $m \in \mathbb{N}$ such that $L \subset H(m, X \setminus \overline{D})$, hence $L \subset H(n, X \setminus \{x_\alpha\})$ for each $\alpha \in \Lambda$, $n \geqslant m$. Thus $L \subset \bigcup_{\alpha \in \Lambda, n < m} H(n, G_\alpha)$, so there are an infinite subset $L' \subset L$ and $n < m$ such that $L' \subset \bigcup_{\alpha \in \Lambda} H(n, G_\alpha)$. By Lemma 2.1(4), L' is discrete in X, a contradiction. □

Lemma 2.3. *Each k-semistratifiable space has a σ-discrete network.*

Proof. Let (X, τ) be a k-semistratifiable space. There is a function $g : \mathbb{N} \times X \to \tau$ such that [4, Theorem 5]

(1) $x \in g(n+1, x) \subset g(n, x)$ for each $n \in \mathbb{N}, x \in X$;
(2) If $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $x_n \to p$ in X, then $y_n \to p$ in X.

Thus if $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for each $n \in \mathbb{N}$, then $x_n \to p$. By the similar proof in [6, Theorem 4.11(v)$\Rightarrow$ (i)], X has a σ-discrete network. □

A space X is said to be a k-space [3] if whenever $K \cap A$ is closed in K for each compact subset K of X, then A is closed in X. Each sequential space is a k-space, and each k-space which each point is a G_δ-set is a sequential space [9].

Lemma 2.4. [8] *Each k-semistratifiable k-space is a hereditarily meta-Lindelöf space.*

k-semistratifiable spaces are preserved by closed mappings [5]. The following is a closed mapping theorem on k-semistratifiable spaces about Question 1.2 and Theorem 1.3.

Theorem 2.5. *Let $f : X \to Y$ be a closed mapping, where X is a k-semistratifiable k-space. Then $\partial f^{-1}(y)$ is Lindelöf(resp. compact) for each $y \in Y$ if Y contains no closed copy of S_{ω_1}(resp. S_ω).*

Proof. For each $y \in Y$, put

$$A = \{x \in \partial f^{-1}(y) : \textit{there is a sequence in } X \setminus f^{-1}(y) \textit{ converging to } x\}.$$

Claim 1: $\overline{A} = \partial f^{-1}(y)$.

If not, let $B = f^{-1}(y) \setminus \overline{A}$, and $C = \partial f^{-1}(y) \setminus \overline{A}$. Then $\emptyset \neq C \subset B$ and B is a sequentially open set of X. In fact, let S be a sequence in X, which converges to a point $x \in B$. If $x \in \mathrm{int}(f^{-1}(y))$, then S is eventually in B. If $x \in C$, then $\overline{A} \cup (X \setminus f^{-1}(y))$ contains no subsequence of S, and so S is eventually in B. And because X is a k-space and each point of X is a G_δ-set, X is a sequential space. Thus B is open in X. Therefore $B \subset \mathrm{int}(f^{-1}(y))$, and $C = C \cap \mathrm{int}(f^{-1}(y)) = \emptyset$, a contradiction.

(1) Suppose Y contains no closed copy of S_{ω_1}. Then we have the following Claim 2.

Claim 2: A is an $\aleph_1$-compact subset of X.

If A is not $\aleph_1$-compact, then X has an uncountable relatively discrete subset $D = \{x_\alpha : \alpha < \omega_1\}$, which is closed discrete in A. Thus $D = \overline{D} \cap A$, and there is a disjoint family $\{U_\alpha : \alpha < \omega_1\}$, which satisfies Lemma 2.2. For each $\alpha < \omega_1$ and $y_\alpha \in U_\alpha \setminus f^{-1}(y)$, by Lemma 2.2, $\{y_\alpha : \alpha < \omega_1\}$ is closed discrete in X. In fact, if $\{y_\alpha : \alpha < \omega_1\}$ contains a sequence converging to a point $x \in \overline{D}$, then $x \in D$. Thus there is $\beta < \omega_1$ such that $x = x_\beta \in U_\beta$. Hence there exist infinitely many y_α's with $y_\alpha \in U_\beta$, a contradiction. Now, it is not difficult to see that for each $\alpha < \omega_1$, there is a sequence L_α in $X \setminus f^{-1}(y)$ such that $L_\alpha \to x_\alpha$, and $\{y\} \cup (\cup\{f(L_\alpha) : \alpha < \omega_1\})$ is a closed copy of S_{ω_1}. This is a contradiction. Hence Claim 2 holds.

A has a σ-discrete network by Lemma 2.3. Thus A has a countable network by Claim 2, so A is separable. Then $\overline{A} = \partial f^{-1}(y)$ is separable by Claim 1. Since a separable meta-Lindelöf space is Lindelöf, $\partial f^{-1}(y)$ is a Lindelöf space by Lemma 2.4.

(2) Suppose Y contains no closed copy of S_ω. By the proof analogous to (1), A is countably compact. Each countably compact k-semistratifiable space is compact by Lemma 2.3, and so $\partial f^{-1}(y) = \overline{A} = A$ is compact.

This completes the proof of Theorem 2.5. □

Remark 2.6. There exists a closed mapping $f : X \to Y$, where X is a Moore space(hence, a k-and σ-space), Y contains no closed copy of S_ω, but f is not peripherally Lindelöf.

Let X be the Isbell-Mrówka space $\psi(\mathbb{N})$(see [2, Example 4.4]), and Y be the convergent sequence $\mathbb{S}_1 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual topology. Then X is a Moore space, and Y contains no closed copy of S_ω. Define a mapping $f : \psi(\mathbb{N}) \to \mathbb{S}_1$ by $f(\psi(\mathbb{N}) \setminus \mathbb{N}) = \{0\}$ and $f(n) = 1/n$ for each $n \in \mathbb{N}$. Thus f is a closed mapping, and $\partial f^{-1}(0) = \psi(\mathbb{N}) \setminus \mathbb{N}$ is not Lindelöf.

The following Remark indicates that k-ness of X in Theorem 2.5 is essential. A space X is said to be an $\aleph$-space if it is a regular space with a σ-locally finite k-network [6].

Remark 2.7. There exists a closed mapping $f : X \to Y$, where X is an $\aleph$-space(hence, a k-semistratifiable space), Y contains no closed copy of S_ω, but f is not peripherally Lindelöf.

Let $S_2 = \{0\} \cup (\bigcup_{i \in \mathbb{N}} X_i)$, where $X_i = \{1/i\} \cup \{1/i + 1/j^2 : j \geqslant i\}$. The S_2 be endowed the Arens topology [3, Example 1.6.19]. For each $\alpha < \omega_1$, put $X_\alpha = S_2 \setminus \{1/n : n \in \mathbb{N}\}$. Then X_α is a paracompact $\aleph$-space. Let $X = \bigoplus_{\alpha < \omega_1} X_\alpha$, and A be the set of all accumulation points in X. Then A is closed in X. Put $Y = X/A$, and let $f : X \to Y$ be a natural quotient mapping. Then f is a closed mapping, and so f is a compact-covering mapping. Since each compact subset of X is finite, each compact subset of Y is finite. Thus Y contains no closed copy of S_ω. But $\partial f^{-1}([A]) = A$ is not Lindelöf.

3. Related Results

In this section, we discuss closed mapping theorems on $k\beta^+$-spaces about Question 1.5.

Definition 3.1. Let (X, τ) be a space, and $g : \mathbb{N} \times X \to \tau$ a function satisfied $x \in g(n+1, x) \subset g(n, x)$ for each $n \in \mathbb{N}, x \in X$. Consider the following conditions on g.

(1) If K is compact in X and $K \cap g(n, y_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\{y_n\}$ has a cluster point in X.

(2) If $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\{x_n\}$ has a cluster point, then $\{y_n\}$ has a cluster point in X;
(3) If $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and each subsequence of $\{x_n\}$ has a cluster point, then $\{y_n\}$ has a cluster point in X;

X is called a $k\beta$-space [17] if there is a function g satisfying the condition (1). X is called a *wcc*-space [18] if there is a function g satisfying the condition (2). X is called a $k\beta^+$-space if there is a function g satisfying the condition (3).

Obviously, *wcc*-spaces $\Rightarrow$ $k\beta^+$-spaces $\Rightarrow$ $k\beta$-spaces. It is easy to check that stratifiable spaces are *wcc*-spaces [18], and k-semistratifiable spaces are $k\beta$-spaces [17]. We don't know whether there is a regular k-semistratifiable space which is not a *wcc*-space.

Lemma 3.2. *Every regular k-semistratifiable space is a $k\beta^+$-space.*

Proof. Let (X, τ) be a regular k-semistratifiable space. By [4, Theorem 5], there is a function $g : \mathbb{N} \times X \to \tau$ such that

(1) $x \in g(n+1, x) \subset g(n, x)$ for each $n \in \mathbb{N}, x \in X$;
(2) If $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $x_n \to p$ in X, then $y_n \to p$ in X.

We shall show that the g satisfies the Definition 3.1(3). Let $\{x_n\}, \{y_n\}$ be two sequences in X such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Suppose that each subsequence of $\{x_n\}$ has a cluster point in X, and let p be a cluster point of $\{x_n\}$. Since X is a regular space, there is a sequence $\{G_n\}$ of open subsets of X such that $\{p\} = \bigcap_{n\in\mathbb{N}} G_n$ and $\overline{G}_{n+1} \subset G_n$ for each $n \in \mathbb{N}$. Then there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \in G_i$ for each $i \in \mathbb{N}$. If $q \in X$ is a cluster point of the sequence $\{x_{n_i}\}$, $q \in \bigcap_{i\in\mathbb{N}} \overline{G}_i = \{p\}$, thus p is the unique cluster point of the sequence $\{x_{n_i}\}$, hence $x_{n_i} \to p$, then $y_{n_i} \to p$. Therefore, X is a $k\beta^+$-space. □

Let $f : X \to Y$ be a mapping. f is called countably compact-covering if for each countably compact subset K in Y, their exists a countably compact subset L in X such that $f(L) \supset K$; f is called quasi-perfect [3] if f is closed and $f^{-1}(y)$ is countably compact for each $y \in Y$.

Theorem 3.3. *Let $f : X \to Y$ be a closed mapping. If X is a $k\beta^+$-space, then Y is a $k\beta^+$-space and f is countably compact-covering.*

Proof. Suppose g is the function on $\mathbb{N} \times X$ satisfying the condition (3) in Definition 3.1. For each $A \subset X, n \in \mathbb{N}$, denoted $\bigcup_{x\in A} g(n, x)$ by $g(n, A)$. Let $h(n, y) = Y \setminus f(X \setminus g(n, f^{-1}(y)))$ for each $n \in \mathbb{N}$ and $y \in Y$. Then $h(y, n)$ is open in Y and $y \in h(n+1,y) \subset h(n,y)$. Assume that $z_n \in h(n,y_n)$ for each $n \in \mathbb{N}$ and any subsequence of $\{z_n\}$ has a cluster point in Y. Then $f^{-1}(z_n) \subset g(n, f^{-1}(y_n))$. For each $n \in \mathbb{N}$, choose $a_n \in f^{-1}(z_n)$, and then there exists $b_n \in f^{-1}(y_n)$ satisfying $a_n \in g(n, b_n)$.

Case1: $\{z_n : n \in \mathbb{N}\}$ is a finite set.

Without loss of generality, suppose $z_n = z \in Y$ for each $n \in \mathbb{N}$. Then pick $a_n = a \in X$ for each $n \in \mathbb{N}$. So $a \in g(n, b_n)$, and $\{b_n\}$ has a cluster point in X, thus $\{y_n\}$ has a cluster point in Y by the continuity of f.

Case 2: $\{z_n : n \in \mathbb{N}\}$ is an infinite set.

We may assume that $z_n \neq z_m$ if $n \neq m \in \mathbb{N}$. Each subsequence of $\{a_n\}$ has a cluster point in X because each subsequence of $\{z_n\}$ has a cluster point in Y and f is closed. Thus $\{b_n\}$ has a cluster point in X, and $\{y_n\}$ has a cluster point in Y.

In a word, Y is a $k\beta^+$-space.

Assume K is a countably compact subset of Y. Pick $x_y \in f^{-1}(y)$ for each $y \in K$. Put $E = \{x_y : y \in K\}$. Then $f(\overline{E}) = \overline{f(E)} = \overline{K} \supset K$. We assert that $\overline{E}$ is countably compact in X. Let $\{x_n\}$ be a sequence in $\overline{E}$ with $x_n \neq x_m$ when $n \neq m$. For each $n \in \mathbb{N}$, $E \cap g(n, x_n) \neq \emptyset$, and choose $z_n \in E \cap g(n, x_n)$.

(1) There is a $p \in X$ such that $z_n = p$ for each $n \in \mathbb{N}$. Then $\{x_n\}$ has a cluster point in $\overline{E}$.

(2) Suppose $z_n \neq z_m$ when $n \neq m \in \mathbb{N}$. Since $f_{|E} : E \to K$ is an injective mapping and K is countably compact, each subsequence of $\{f(z_n)\}$ has a cluster point in K. Then each subsequence of $\{z_n\}$ has a cluster point in X. Hence $\{x_n\}$ has a cluster point in $\overline{E}$.

Therefore, f is a countably compact-covering mapping. □

Corollary 3.4. [7] *Each closed mapping from a regular k-semistratifiable space is compact-covering.*

Theorem 3.5. *Let $f : X \to Y$ be a quasi-perfect mapping. If Y is a $k\beta^+$-space, then X is a $k\beta^+$-space.*

Proof. Let g be the function on $\mathbb{N} \times Y$ satisfying the condition (3) in Definition 3.1 for a $k\beta^+$-space Y. Definite $h(n, x) = f^{-1}(g(n, f(x)))$ for each $n \in \mathbb{N}$ and $x \in X$. Then $h(n, x)$ is open in X and $x \in h(n + 1, x) \subset h(n, x)$. Assume that $x_n \in h(n, z_n)$ for each $n \in \mathbb{N}$ and any subsequence of $\{x_n\}$ has a cluster point in X. Then any subsequence of $\{f(x_n)\}$ has a cluster point in Y. Since $f(x_n) \in g(n, f(z_n))$, $\{f(z_n)\}$ has a cluster point in Y.

(1) Suppose that there is $y_0 \in Y$ with $f(z_n) = y_0$ for each $n \in \mathbb{N}$. Then $z_n \in f^{-1}(y_0)$. Since $f^{-1}(y_0)$ is countably compact, $\{z_n\}$ has a cluster point in X.

(2) Suppose $f(z_n) \neq f(z_m)$ when $n \neq m \in \mathbb{N}$. Then $\{z_n\}$ has a cluster point in X.

Hence, X is a $k\beta^+$-space. □

We don't know whether there is a $k\beta$-space which is not a $k\beta^+$-space. The authors thank to the referee of this paper for his following result related with the question. A space X is isocompact [1] if every closed countably compact set in X is compact.

Theorem 3.6. *Let X be a $k\beta$-space. If X satisfies one of the conditions below, then X is a $k\beta^+$-space.*

(1) *a regular space whose points are G_δ-sets;*
(2) *a k-space;*
(3) *an isocompact space which is normal or countably paracompact.*

Indeed, let $\{x_n\}, \{y_n\}$ be two sequences in a $k\beta$-space X such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, and each subsequence of $\{x_n\}$ has a cluster point in X. Denote $L = \{x_n : n \in \mathbb{N}\}$. For (1), $\{x_n\}$ has a convergent subsequence by the similar proof in Lemma 3.2. For (2), we can assume that L is not closed in X. Thus, L has a subsequence contained in a compact set in X. For (3), since L is relatively rightly compact [12](i. e., whenever $\mathcal{U}$ is a locally finite collection of open subsets of X, L meets at most finitely many $U \in \mathcal{U}$), cl$L$ is countably compact by [12, Proposition 3.1]. Hence $\{y_n\}$ has a cluster point in X. Thus X is a $k\beta^+$-space.

By Theorems 3.3 and 3.6, the answer of Question 1.5 is positive if the domain satisfies one of the conditions in Theorem 3.6.

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