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On Ponomarev-Systems (*).

YING GE - SHOU LIN

Sunto. – *In questo lavoro vengono studiate le relazioni fra mappe e famiglie di sottoinsiemi nei sistemi di Ponomarev, e si ottengono i seguenti risultati.* (1) f *è una "sequence-covering" (risp. una "1-sequence-covering") mappa se e solo se* $\mathcal{P}$ *è una csf rete (risp. una snf rete) di* X *per un sistema di Ponomarev* $(f, M, X, \mathcal{P})$; (2) f *è una "sequence-covering" (risp. una "1-sequence-covering") mappa se e solo se ogni* $\mathcal{P}_n$ *è un cs ricoprimento (risp. un wsn ricoprimento) di* X *per un sistema di Ponomarev* $(f, M, X, \{\mathcal{P}_n\})$. *Come applicazione di questi risultati vengono discusse alcune relazioni fra "sequence-covering" mappe e "1-sequence-covering" mappe, e si fornisce la risposta a una domanda posta da S. Lin.*

Summary. – *In this paper the relations of mappings and families of subsets are investigated in Ponomarev-systems, and the following results are obtained.* (1) f *is a sequence-covering (resp. 1-sequence-covering) mapping iff* $\mathcal{P}$ *is a csf-network (resp. snf-network) of* X *for a Ponomarev-system* $(f, M, X, \mathcal{P})$; (2) f *is a sequence-covering (resp. 1-sequence-covering) mapping iff every* $\mathcal{P}_n$ *is a cs-cover (resp. wsn-cover) of* X *for a Ponomarev-system* $(f, M, X, \{\mathcal{P}_n\})$. *As applications of these results, some relations between sequence-covering mappings and 1-sequence-covering mappings are discussed, and a question posed by S. Lin is answered.*

1. – Introduction.

In 1960, V. I. Ponomarev [11] proved that every first countable space can be characterized as an open image of a subspace of a Baire's zero-dimensional space. Recently S. Lin [6] generalized the "Ponomarev's method" to established two systems $(f, M, X, \mathcal{P})$ and $(f, M, X, \{\mathcal{P}_n\})$, which are called Ponomarev-systems [9, 13]. The following results have be obtained [6, 13].

THEOREM 1.1. – *The following hold for a Ponomarev-system* $(f, M, X, \mathcal{P})$.

(1) *If* $\mathcal{P}$ *is a point-finite (resp. point-countable) network of* X, *then* f *is a compact mapping (resp. s-mapping).*

(2) *If* $\mathcal{P}$ *is a point-countable cs-network of* X, *then* f *is a sequence-covering, s-mapping.*

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(3) *If $\mathcal{P}$ is a point-countable sn-network of X, then f is a* 1*-sequence-covering, s-mapping.*

THEOREM 1.2. – *The following hold for a Ponomarev-system* $(f, M, X, \{\mathcal{P}_n\})$.

(1) *If every* $\mathcal{P}_n$ *is a point-finite (resp. point-countable) cover of X, then f is a compact mapping (resp. s-mapping).*

(2) *If every* $\mathcal{P}_n$ *is a cs-cover (resp. sn-cover) of X, then f is a sequence-covering mapping (resp.* 1*-sequence-covering mapping).*

Take the above theorems into account, the following question naturally arises.

QUESTION 1.3. – *Can implications in Theorem* 1.1 *and Theorem* 1.2 *be reversed?*

In addition, P. Yan and S. Lin proved that every sequence-covering, compact mapping from a metric space is 1-sequence-covering ([8, Theorem 4.4]. In view of this result, the following question was posed by S. Lin in [6, Question 3.4.3].

QUESTION 1.4. – *Is every sequence-covering, π-mapping from a metric space* 1*-sequence-covering?*

In this paper, we investigate the above two Ponomarev-systems, answer Question 1.3 affirmatively except for 1-sequence-covering mapping, and give two sufficient and necessary conditions such that f is a 1-sequence-covering mapping in two Ponomarev-systems respectively. As some applications of these results, for a Ponomarev-system $(f, M, X, \mathcal{P})$, f is a sequence-covering, s-mapping iff it is a 1-sequence-covering, s-mapping for an sn-first countable space X, and where sn-first countability of X can not be omitted; for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, f is a sequence-covering, s-mapping iff it is a 1-sequence-covering, s-mapping, and where "s" can not be omitted, which answers Question 1.4 negatively.

Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are continuous and onto. Let X be a space and $A \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in A if $\{x_n : n > k\} \cup \{x\} \subset A$ for some $k \in \mathbb{N}$. Let $\mathcal{P}$ be a family of subsets of X and $x \in X$. $st(x, \mathcal{P})$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P \in \mathcal{P} : x \in P\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively. For a sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$ of covers of a space X and a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{\mathcal{P}_n : n \in \mathbb{N}\}$ and $\{P_n : n \in \mathbb{N}\}$ to $\{\mathcal{P}_n\}$ and $\{P_n\}$ respectively. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n), and the n-th coordinate β_n of b is also denoted by $(b)_n$.

2. – On Ponomarev-system $(f, M, X, \mathcal{P})$.

DEFINITION 2.1. – *Let* $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ *be a cover of a space X, where* $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ *is called a network of X[10] if for every* $x \in U$ *with U open in X, there exists* $P \in \mathcal{P}_x$ *such that* $x \in P \subset U$, *where* $\mathcal{P}_x$ *is called a network at x in X.*

DEFINITION 2.2. – *Let* $\mathcal{P}$ *be a network of a space X. Assume that there exists a countable* $\mathcal{P}_x \subset \mathcal{P}$ *such that* $\mathcal{P}_x$ *is a network at x in X for every* $x \in X$. *Put* $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$. *For every* $n \in \mathbb{N}$, *put* $\Lambda_n = \Lambda$ *and endow* Λ_n *a discrete topology. Put* $M = \{b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{P_{\beta_n}\}$ *forms a network at some point* x_b *in* $X\}$, *then M, which is a subspace of the product space* $\Pi_{n \in \mathbb{N}} \Lambda_n$, *is a metric space and* x_b *is unique for every* $b \in M$. *Define* $f : M \to X$ *by* $f(b) = x_b$, *then f is a mapping, and* $(f, M, X, \mathcal{P})$ *is called a Ponomarev-system* [9, 13].

DEFINITION 2.3. – *Let* (X, d) *be a metric space, and let* $f : X \to Y$ *be a mapping. f is called a* π*-mapping* [11] *if for every* $y \in U$ *with U open in Y,* $d(f^{-1}(y), X - f^{-1}(U)) > 0$.

REMARK 2.4. – Recall a mapping $f : X \to Y$ is a compact mapping (resp. s-mapping) if $f^{-1}(y)$ is a compact (resp. separable) subset of X for every $y \in Y$. It is clear that every compact mapping from a metric space is an s- and π-mapping.

DEFINITION 2.5. – *Let* $f : X \to Y$ *be a mapping.*

(1) *f is called a sequence-covering mapping* [12] *if whenever* $\{y_n\}$ *is a convergent sequence in Y there exists a convergent sequence* $\{x_n\}$ *in X with every* $x_n \in f^{-1}(y_n)$;

(2) *f is called a 1-sequence-covering mapping* [8] *if for every* $y \in Y$ *there exists* $x \in f^{-1}(y)$ *such that whenever* $\{y_n\}$ *is a sequence converging to y in Y there exists a sequence* $\{x_n\}$ *converging to x in X with every* $x_n \in f^{-1}(y_n)$.

REMARK 2.6. – (1) Sequence-covering mapping in Definition 2.5(1), which is called sequence-covering mapping in the sense of Siwiec, is different from sequence-covering mapping in the sense of Gruenhage-Michael-Tanaka. G. Gruenhage, E. Michael and Y. Tanaka [3] called a mapping $f : X \to Y$ a sequence-covering mapping if for every sequence S converging to y in Y, there exists a compact subset K of X such that $f(K) = S \cup \{y\}$(also see [7]). In this paper, we deal with sequence-covering mapping in the sense of Siwiec.

(2) Every sequence-covering, compact mapping from a metric space is 1-sequence-covering [8].

DEFINITION 2.7. – *Let X be a space and $x \in X$. A subset P of X is called a sequential neighborhood of x if every sequence converging to x in X is eventually in P.*

DEFINITION 2.8. – *Let $\mathcal{P}$ be a cover of a space X.*

(1) *$\mathcal{P}$ is called a cs-network of X* [4] *if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then $\{x_n : n \geq m\} \cup \{x\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.*

(2) *$\mathcal{P}$ is called a csf-network of X* [2] *if whenever S is a sequence converging to a point x in X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a network at x in X and S is eventually in P for every $P \in \mathcal{P}_S$, where $\mathcal{P}_S$ is called a csf-network for S in X.*

(3) *$\mathcal{P}$ is called an sn-network of X* [8] *if $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$, and $\mathcal{P}_x$ satisfies the following conditions (a), (b) and (c) for every $x \in X$, where $\mathcal{P}_x$ is called an sn-network at x in X.*

(a) *$\mathcal{P}_x$ is a network at x in X.*
(b) *If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.*
(c) *Every element of $\mathcal{P}_x$ is a sequential neighborhood of x.*

If $\mathcal{P}_x$ is also countable for every $x \in X$, then X is called sn-first countable [5, 1]

(4) *$\mathcal{P}$ is called an snf-network of X, if for every $x \in X$, there exists a countable subfamily $\mathcal{P}_x$ of $\mathcal{P}$ satisfying the above conditions (a) and (c), where $\mathcal{P}_x$ is called an snf-network at x in X.*

REMARK 2.9. – The following are clear.

(1) *sn*-networks⇒*cs*-networks.
(2) *snf*-networks⇒*csf*-networks⇒*cs*-networks.
(3) point-countable *cs*-networks⇒*csf*-networks.

LEMMA 2.10. – *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system and let $U = (\Pi_{n \in \mathbb{N}} \Gamma_n) \bigcap M$, where $\Gamma_n \subset \Lambda_n$ for every $n \in \mathbb{N}$. Then $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$ for every $k \in \mathbb{N}$.*

PROOF. – Let $b = (\beta_n) \in U$ and let $k \in \mathbb{N}$. Then $\{P_{\beta_n}\}$ forms a network at $f(b)$ in X and $\beta_k \in \Gamma_k$. So $f(b) \in P_{\beta_k} \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. This proves that $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. □

PROPOSITION 2.11. – *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. Then f is a compact mapping (resp. s-mapping) iff $\mathcal{P}$ is point-finite (point-countable) network of X.*

PROOF. – By Theorem 1.1(1), we only need to prove necessities.

We only give a proof for the parenthetic part. If $\mathcal{P}$ is not point-countable, then, for some $x \in X$, there exists an uncountable subset Γ of Λ such that $\Gamma = \{\beta \in \Lambda : x \in P_\beta\}$. Let $\{P_{\beta_n}\}$ forms a network at x in X. For every $\beta \in \Gamma$, put $c_\beta = (\gamma_n)$, where $\gamma_1 = \beta$, and $\gamma_n = \beta_{n-1}$ for $n > 1$, then $\{P_{\gamma_n}\}$ forms a network at x in X, so $c_\beta \in f^{-1}(x)$. Put $U_\beta = (\{\beta\} \times (\Pi_{n>1}\Lambda_n)) \bigcap M$ for every $\beta \in \Gamma$, then $\{U_\beta : \beta \in \Gamma\}$ covers $f^{-1}(x)$. If not, there exists $c = (a_n) \in f^{-1}(x)$ and $c \notin U_\beta$ for every $\beta \in \Gamma$, so $a_1 \notin \Gamma$. Thus $x \notin P_{a_1}$ from construction of Γ. But $x = f(c) \in P_{a_1}$ from Lemma 2.10. This is a contradiction. Thus $\{U_\beta : \beta \in \Gamma\}$ is an uncountable open cover of $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence f is not an s-mapping. □

Now we investigate reversibility of Theorem 1.1(2),(3). At first, we give sufficient and necessary conditions such that f is sequence-covering and 1-sequence-covering respectively for a Ponomarev-system $(f, M, X, \mathcal{P})$.

LEMMA 2.12. – *Let $f : X \to Y$ be a mapping, and $\{y_n\}$ be a sequence converging to y in Y. If $\{B_n\}$ is a decreasing network at some $x \in f^{-1}(y)$ in X, and $\{y_n\}$ is eventually in $f(B_n)$ for every $n \in N$, then there is a sequence $\{x_n\}$ converging to x such that every $x_n \in f^{-1}(y_n)$.*

PROOF. – Let $\{B_n\}$ be a decreasing network at some $x \in f^{-1}(y)$ in X, and let $\{y_n\}$ be eventually in $f(B_k)$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $y_n \in f(B_k)$ for $n > n_k$, so $f^{-1}(y_n) \bigcap B_k \neq \emptyset$ for every $n > n_k$. Without loss of generality, we can assume $1 < n_k < n_{k+1}$. For every $n \in \mathbb{N}$, pick $x_n \in f^{-1}(y_n)$ if $n < n_1$, and pick $x_n \in f^{-1}(y_n) \cap B_k$ if $n_k \leq n < n_{k+1}$, then $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$. It is not difficult to prove that $\{x_n\}$ converges to x. □

THEOREM 2.13. – *The following hold for a Ponomarev-system $(f, M, X, \mathcal{P})$.*

(1) *f is a sequence-covering mapping iff $\mathcal{P}$ is a csf-network of X.*
(2) *f is a 1-sequence-covering mapping iff $\mathcal{P}$ is an snf-network of X.*

PROOF. – (1) Sufficiency. Let $\mathcal{P}$ be a *csf*-network of X, and let $S = \{x_n\}$ be a sequence converging to x in X. Then there exists a countable subfamily $\mathcal{P}_S = \{P_{\beta_n}\}$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a *csf*-network for S in X. It is clear that $\{x_n\}$ is eventually in $\bigcap_{i \leq n} P_{\beta_i}$ for every $n \in \mathbb{N}$. Put $b = (\beta_n)$, then $b \in f^{-1}(x)$. For every $n \in \mathbb{N}$, put $B_n = \{(\gamma_i) \in M : \gamma_i = \beta_i \text{ for } i \leq n\}$. Then $\{B_n\}$ is a decreasing neighborhood base at b in Y. It is not difficulty to prove that $f(B_n) = \bigcap_{i \leq n} P_{\beta_i}$. Thus $\{x_n\}$ is eventually in $f(B_n)$ for every $n \in \mathbb{N}$. By Lemma 2.12, there exists a sequence $\{b_n\}$ converging to b in M with every $b_n \in f^{-1}(x_n)$. This proves that f is sequence-covering.

Necessity. Let f be sequence-covering. Suppose that $S = \{x_n\}$ is a sequence converging to x in X. There exists a sequence $\{b_n\}$ converging to b in M such that $f(b_n) = x_n$ for every $n \in \mathbb{N}$. Let $b = (\beta_k) \in (\Pi_{k\in\mathbb{N}}A_k)\bigcap M$. Then $\{P_{\beta_k} : k \in \mathbb{N}\} \subset \mathcal{P}$ is a network at x in X. For every $k \in \mathbb{N}$, Put $B = ((\Pi_{i<k}A_i) \times \{\beta_k\} \times (\Pi_{i>k}A_i))\bigcap M$, then B is an open neighborhood of b in M. Thus $\{b_n\}$ is eventually in B, and so $\{x_n\}$ is eventually in $f(B)$. Since $f(B) \subset P_{\beta_k}$ from Lemma 2.10, so $\{x_n\}$ is eventually in P_{β_k}, hence $\{P_{\beta_k} : k \in \mathbb{N}\} \subset \mathcal{P}$ is a csf-network for S in X. This proves that $\mathcal{P}$ is a csf-network of X.

(2) Sufficiency. Let $\mathcal{P}$ be an snf-network of X. For every $x \in X$, let $\{P_{\beta_n}\}$ be an snf-network at x in X. Then $\beta_n \in A_n$ and P_{β_n} is a sequential neighborhood of x for every $n \in \mathbb{N}$. Put $b = (\beta_n)$, then $b \in f^{-1}(x)$. For every $n \in \mathbb{N}$, put $B_n = \{(\gamma_i) \in M : \gamma_i = \beta_i \text{ for } i \leq n\}$. Then $\{B_n\}$ is a decreasing neighborhood base at b in Y. It is not difficulty to prove that $f(B_n) = \bigcap_{i\leq n} P_{\beta_i}$. Note that the intersection of finite sequential neighborhoods of x is sequential neighborhood of x in X, so $f(B_n)$ is a sequential neighborhood of x in X. Let $\{x_i\}$ be a sequence in X converging to x. Then there exists a sequence $\{b_n\}$ converging to b in M with every $b_n \in f^{-1}(x_n)$ from Lemma 2.12. This proves that f is 1-sequence-covering.

Necessity. Let f be 1-sequence-covering. If $x \in X$, then there exists $b \in f^{-1}(x)$ such that whenever $\{x_i\}$ is a sequence converging to x in X there exists a sequence $\{b_i\}$ converging to b in M with every $b_i \in f^{-1}(x_i)$. Let $b = (\beta_n) \in M \subset \Pi_{n\in\mathbb{N}}A_n$. Then $\{P_{\beta_n}\} \subset \mathcal{P}$ is a network at x in X. It suffices to prove that P_{β_n} is a sequential neighborhood of x for every $n \in \mathbb{N}$. Whenever $\{y_i\}$ is a sequence converging to x in X there exists a sequence $\{c_i\}$ converging to b in M with every $c_i \in f^{-1}(y_i)$. Put $U_b = ((\Pi_{i<n}A_i) \times \{\beta_n\} \times (\Pi_{i>n}A_i))\bigcap M$, then U_b is an open neighborhood of b in M. So sequence $\{c_i\}$ is eventually in U_b, hence sequence $\{y_i\}$ is eventually in $f(U_b)$. $f(U_b) \subset P_{\beta_n}$ from Lemma 2.10, so $\{y_i\}$ is eventually in P_{β_n}. This proves that P_{β_n} is a sequential neighborhood of x. □

The following corollary is obtained immediately from Remark 2.9, Proposition 2.11 and Theorem 2.13.

COROLLARY 2.14. – *The following hold for a Ponomarev-system* $(f, M, X, \mathcal{P})$.

(1) f *is a sequence-covering, s-mapping iff* $\mathcal{P}$ *is a point-countable cs-network of* X.

(2) f *is a* 1*-sequence-covering, s-mapping iff* $\mathcal{P}$ *is a point-countable snf-network of* X.

REMARK 2.15. – Whether $\mathcal{P}$ is a cs-network of X iff f is sequence-covering for a Ponomarev-system $(f, M, X, \mathcal{P})$? The answer is negative. In fact, let X be the sequential fan space S_ω. Put $\mathcal{P} = \{U \subset X : U \textit{ is open in } X\} \bigcup \{\{x\} : x \in X\}$, then $\mathcal{P}$ is a cs-network of X and $(f, M, X, \mathcal{P})$ is a Ponomarev-system, but $\mathcal{P}$ is not a csf-network of X.

EXAMPLE 2.16. – There exists a Ponomarev-system $(f, M, X, \mathcal{P})$ such that f is a 1-sequence-covering, s-mapping, but $\mathcal{P}$ is not an sn-network. So the implication in Theorem 1.1(3) can not be reversed.

PROOF. – Let X be a non-discrete space with a point-countable sn-network $\mathcal{P}'$. Put $\mathcal{P} = \mathcal{P}' \bigcup \{\{x\} : x \in X\}$, then $\mathcal{P}$ is a point-countable snf-network, and is not an sn-network. Consider Ponomarev-system $(f, M, X, \mathcal{P})$. Then f is a 1-sequence-covering, s-mapping from Corollary 2.14. □

PROPOSITION 2.17. – *Let X be an sn-first countable space. If $\mathcal{P}$ is point-countable, then $\mathcal{P}$ is a cs-network of X iff it is an snf-network of X.*

PROOF. – We only need to prove necessity. Let $\mathcal{P}$ be a point-countable cs-network of X. For every $x \in X$, let $\{F_n : n \in \mathbb{N}\}$ be an sn-network at x in X. Let $\mathcal{P}_x = \{P \in \mathcal{P} : F_n \subset P \text{ for some } n \in \mathbb{N}\}$. Then every element of $\mathcal{P}_x$ is a sequential neighborhood of x, and $\mathcal{P}_x$ is countable. It suffices to prove that $\mathcal{P}_x$ is a network at x in X. To show this, let U be an open neighborhood of x in X. Then there exists $P \in \mathcal{P}_x$ such that $P \subset U$. Otherwise, let $\{P \in \mathcal{P} : x \in P \subset U\} = \{P_m : m \in \mathbb{N}\}$. Then for every $n, m \in \mathbb{N}$, $F_n \not\subset P_m$, so choose $x_{n,m} \in F_n - P_m$. For $n \geq m$, let $x_{n,m} = y_k$, where $k = m + n(n-1)/2$. Then the sequence $\{y_k\}$ converges to x. Thus, there exist $m, i \in \mathbb{N}$ such that $\{y_k : k \geq i\} \cup \{x\} \subset P_m \subset U$. Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m$. This is a contradiction. Thus $\mathcal{P}_x$ is a network at x in X. □

THEOREM 2.18. – *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. If X is sn-first countable, then the following are equivalent.*

(1) *f is a sequence-covering, s-mapping;*
(2) *f is a 1-sequence-covering, s-mapping.*

PROOF. – Consider the following conditions.

(3) $\mathcal{P}$ is a point-countable cs-network of X;
(4) $\mathcal{P}$ is a point-countable snf-network of X.

(1)⇔(3) and (2)⇔(4) from Corollary 2.14. (3)⇔(4) from Proposition 2.17. Thus (1)⇔(2). □

REMARK 2.19. – S. Lin gave an example to show that a sequence-covering, s-image of a metric space need not to be a 1-sequence-covering image of a metric space[6, Example 3.4.7(7)], so sn-first countability of X in Theorem 2.18 can not be omitted.

3. – On Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.

DEFINITION 3.1. – *Let* $\{\mathcal{P}_n\}$ *be a sequence of covers of a space* X. $\{\mathcal{P}_n\}$ *is called a point-star network of* X[9] *if* $\{st(x, \mathcal{P}_n)\}$ *is a network at* x *in* X *for every* $x \in X$.

DEFINITION 3.2. – *Let* $\{\mathcal{P}_n\}$ *is a point-star network of a space* X. *For every* $n \in \mathbb{N}$, *put* $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$ *and endow* Λ_n *a discrete topology. Put* $M = \{b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n : \{P_{\beta_n}\}$ *forms a network at some point* x_b *in* $X\}$, *then* M, *which is a subspace of the product space* $\Pi_{n\in\mathbb{N}}\Lambda_n$, *is a metric space and* x_b *is unique for every* $b \in M$. *Define* $f : M \to X$ *by* $f(b) = x_b$, *then* f *is a mapping, and* $(f, M, X, \{\mathcal{P}_n\})$ *is called a Ponomarev-system* [9, 13].

REMARK 3.3. – $f : M \to X$ is a π-mapping for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$[9, 13].

DEFINITION 3.4. – *Let* $\mathcal{P}$ *be a cover of a space* X.

(1) $\mathcal{P}$ *is called a cs-cover of* X[14] *if for every convergent sequence* S *in* X, *there exists* $P \in \mathcal{P}$ *such that* S *is eventually in* P;

(2) $\mathcal{P}$ *is called an sn-cover of* X[8] *if every element of* $\mathcal{P}$ *is a sequential neighborhood of some point in* X, *and for every* $x \in X$, *there exists* $P \in \mathcal{P}$ *such that* P *is a sequential neighborhood of* x;

(3) $\mathcal{P}$ *is called a wsn-cover of* X *if for every* $x \in X$, *there exists* $P \in \mathcal{P}$ *such that* P *is a sequential neighborhood of* x.

REMARK 3.5. – It is clear that "sn-cover"$\Rightarrow$ "wsn-cover"$\Rightarrow$ "cs-cover".

The proof of the following lemma is as to that of Lemma 2.10, we omit it.

LEMMA 3.6. – *Let* $(f, M, X, \{\mathcal{P}_n\})$ *be a Ponomarev-system and let* $U = (\Pi_{n\in\mathbb{N}}\Gamma_n)\bigcap M$, *where* $\Gamma_n \subset \Lambda_n$ *for every* $n \in \mathbb{N}$. *Then* $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$ *for every* $k \in \mathbb{N}$.

THEOREM 3.7. – *The following hold for a Ponomarev-system* $(f, M, X, \{\mathcal{P}_n\})$.

(1) f *is a compact mapping (resp. s-mapping) iff* $\mathcal{P}_m$ *is a point-finite (resp. point-countable) cover of* X *for every* $m \in \mathbb{N}$.

(2) *f is a sequence-covering mapping iff $\mathcal{P}_m$ is a cs-cover of X for every $m \in \mathbb{N}$.*

PROOF. – By Theorem 1.2, we only need to prove necessities of (1) and (2). Let $m \in \mathbb{N}$.

(1) We only give a proof for the parenthetic part. If $\mathcal{P}_m$ is not point-countable, then, for some $x \in X$, there exists an uncountable subset Γ_m of Λ_m such that $\Gamma_m = \{\beta \in \Lambda_m : x \in P_\beta\}$. For every $\beta \in \Gamma_m$, put $U_\beta = ((\Pi_{n<m}\Lambda_n) \times \{\beta\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$. Then $\{U_\beta : \beta \in \Gamma_m\}$ covers $f^{-1}(x)$. If not, there exists $c = (\gamma_n) \in f^{-1}(x)$ and $c \notin U_\beta$ for every $\beta \in \Gamma_m$, so $\gamma_m \notin \Gamma_m$. Thus $x \notin P_{\gamma_m}$ from construction of Γ_m. But $x = f(c) \in P_{\gamma_m}$ from Lemma 3.6. This is a contradiction. Thus $\{U_\beta : \beta \in \Gamma_m\}$ is an uncountable open cover of $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence f is not an s-mapping.

(2) Let f be sequence-covering and $m \in \mathbb{N}$. If $\{x_i\}$ be a sequence converging to x in X, then there exists a sequence $\{b_i\}$ converging to b in M such that $f(b_i) = x_i$ for every $i \in \mathbb{N}$. Let $b = (\beta_n) \in M$. We claim that sequence $\{x_i\}$ is eventually in P_{β_m}. In fact, put $U = ((\Pi_{n<m}\Lambda_n) \times \{\beta_m\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$, then U is an open neighborhood of b in M. So sequence $\{b_i\}$ is eventually in U, hence sequence $\{x_i\}$ is eventually in $f(U)$. $f(U) \subset P_{\beta_m}$ from Lemma 3.6, so $\{x_i\}$ is eventually in $P_{\beta_m} \in \mathcal{P}_m$. This proves that $\mathcal{P}_m$ is a cs-cover of X. □

The following examples show that "f is 1-sequence-covering"$\not\Rightarrow$ "every $\mathcal{P}_n$ is an sn-cover" for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.

EXAMPLE 3.8. – There exists a point-star network $\{\mathcal{P}_n\}$ consisting of point-finite cs-covers of a space X such that every $\mathcal{P}_n$ is not an sn-cover of X.

PROOF. – Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ endow usual subspace topology of real line $\mathbb{R}$. Put $A_n = \{1/k : k > n\}$, and $\mathcal{P}_n = \{A_n \cup \{0\}\} \bigcup \{\{1/k\} : k = 1, 2, \cdots, n\} \bigcup \{\{0\}\}$ for every $n \in \mathbb{N}$.

Claim 1: $\{\mathcal{P}_n\}$ is a point-star network of X.

If $0 \in U$ with U open in X, then there exists $m \in \mathbb{N}$ such that $A_m \subset U$. It is easy to see that $st(0, \mathcal{P}_m) = A_m \cup \{0\}$, so $0 \in st(0, \mathcal{P}_m) \subset U$. If $1/n \in U$ with U open in X, where $n \in \mathbb{N}$. It is easy to see that $st(1/n, \mathcal{P}_n) = \{1/n\}$, so $1/n \in st(1/n, \mathcal{P}_n) \subset U$. Thus $\mathcal{P}_n$ is a point-star network of X.

Claim 2: $\mathcal{P}_n$ is a point-finite cs-cover of X for every $n \in \mathbb{N}$.

Let $S = \{x_k\}$ be a sequence converging to x in X. If $x = 0$ or $x = 1/m$ with $m > n$, then S is eventually in $A_n \cup \{0\} \in \mathcal{P}_n$. If $x = 1/m$ with $m \leq n$, then S is eventually in $\{1/m\} \in \mathcal{P}_n$. So $\mathcal{P}_n$ is a cs-cover of X. Note that $\mathcal{P}_n$ is a finite cover of X. So $\mathcal{P}_n$ is a point-finite cs-cover of X.

Claim 3: $\mathcal{P}_n$ is not an sn-cover of X for every $n \in \mathbb{N}$.

Note that $\{0\} \in \mathcal{P}_n$, and $\{0\}$ is not a sequential neighborhood of x for every $x \in X$. So $\mathcal{P}_n$ is not an sn-cover of X. □

EXAMPLE 3.9. – There exists a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ such that f is a 1-sequence-covering mapping, but every $\mathcal{P}_n$ is not an sn-cover of X.

PROOF. – Let $\{\mathcal{P}_n\}$ be a point-star network stated in Example 3.8. Then Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ is demanded. In fact, every $\mathcal{P}_n$ is a point-finite cs-cover of X from Claim 2 in Example 3.8, so f is a sequence-covering, compact mapping from Theorem 3.7, hence f is a 1-sequence-covering mapping from Remark 2.6(2). On the other hand, every $\mathcal{P}_n$ is not an sn-cover of X from Claim 3 in Example 3.8. □

The following question is posed by the above example. What is the sufficient and necessary condition such that f is a 1-sequence-covering mapping in a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$? We give an answer to this question.

THEOREM 3.10. – *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then f is a 1-sequence-covering mapping if and only if every $\mathcal{P}_n$ is a wsn-cover of X.*

PROOF. – Sufficiency. Let every $\mathcal{P}_n$ be a wsn-cover of X, and let $x \in X$. For every $n \in \mathbb{N}$, pick $\beta_n \in \Lambda_n$ such that P_{β_n} is a sequential neighborhood of x. Then $\{P_{\beta_n}\}$ forms a network at x in X. Put $b = (\beta_n)$, then $b \in f^{-1}(x)$. Let $\{x_i\}$ be a sequence in X converging to x. Then $\{x_i\}$ is eventually in P_{β_n} for every $n \in \mathbb{N}$. Let $i \in \mathbb{N}$. For every $n \in \mathbb{N}$, if $x_i \in P_{\beta_n}$, put $\beta_{i,n} = \beta_n$; if $x_i \notin P_{\beta_n}$, pick $\beta_{i,n} \in \Lambda_n$ such that $x_i \in P_{\beta_{i,n}}$. Then $\{P_{\beta_{i,n}}\}_{n\in\mathbb{N}}$ forms a network at x_i in X. Put $b_i = (\beta_{i,n})_{n\in\mathbb{N}}$, then $b_i \in f^{-1}(x_i)$. We only need to prove that sequence $\{b_i\}$ converges to b in M. It suffices to prove that for every $n \in \mathbb{N}$, sequence $\{\beta_{i,n}\}_{i\in\mathbb{N}}$ converges to β_n in Λ_n. Let $n \in \mathbb{N}$. $\{x_i\}$ is eventually in P_{β_n}, so there exists $i_n \in \mathbb{N}$ such that $x_i \in P_{\beta_n}$ for $i > i_n$, and so $\beta_{i,n} = \beta_n$ for $i > i_n$. Thus sequence $\{\beta_{i,n}\}_{i\in\mathbb{N}}$ converges to β_n in Λ_n.

Necessity. Let f be 1-sequence-covering. If $x \in X$, then there exists $b \in f^{-1}(x)$ such that whenever $\{x_n\}$ is a sequence converging to x in X there exists a sequence $\{b_n\}$ converging to b in M with every $b_n \in f^{-1}(x_n)$. Let $b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n$. It suffices to prove that $P_{\beta_m} \in \mathcal{P}_m$ is a sequential neighborhood of x for every $m \in \mathbb{N}$. Whenever $\{y_n\}$ is a sequence converging to x in X there exists a sequence $\{c_n\}$ converging to b in M with every $c_n \in f^{-1}(y_n)$. Put $U_b = ((\Pi_{n<m}\Lambda_n) \times \{\beta_m\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$, then U_b is an open neighborhood of b in M. So sequence $\{c_n\}$ is eventually in U_b, hence sequence $\{y_n\}$ is eventually in $f(U_b)$. $f(U_b) \subset P_{\beta_m}$ from Lemma 3.6, so $\{y_n\}$ is eventually in P_{β_m}. This proves that $P_{\beta_m} \in \mathcal{P}_m$ is a sequential neighborhood of x. □

DEFINITION 3.11. – *A point-star network $\{\mathcal{P}_n\}$ of X is called a point-star sn-network of X if $\{st(x, \mathcal{P}_n)\}$ is an sn-network at x in X for every $x \in X$.*

REMARK 3.12. – A space X with a point-star network consisting of cs-covers has a point-star sn-network from [1, Theorem 2.5], and so is sn-first countable from [1, Remark 1.12].

PROPOSITION 3.13. – *Let X be an sn-first countable space. If $\mathcal{P}$ is point-countable, then $\mathcal{P}$ is a cs-cover of X iff it is a wsn-cover of X.*

PROOF. – We only need to prove necessity. Let $\mathcal{P}$ be a point-countable cs-cover of X. For every $x \in X$, put $(\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}$, and let $\{F_n : n \in \mathbb{N}\}$ be an sn-network at x in X. Then $F_n \subset P_m$ for some $n, m \in \mathbb{N}$ by the proof of Proposition 2.17. Hence $\mathcal{P}$ is a wsn-cover of X.

THEOREM 3.14. – *The following are equivalent for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.*

(1) *f is a sequence-covering, s-mapping;*

(2) *f is a 1-sequence-covering, s-mapping.*

PROOF. – Consider the following conditions.

(3) $\mathcal{P}_n$ is a point-countable cs-cover of X for every $n \in \mathbb{N}$;

(4) $\mathcal{P}_n$ is a point-countable wsn-cover of X for every $n \in \mathbb{N}$.

(1)⇔(3) and (2)⇔(4) from Theorem 3.7 and Theorem 3.10 respectively. If one of (3) and (4) holds, then X is sn-first countable from Remark 3.12, so (3)⇔(4) from Proposition 3.13. Thus (1)⇔(2). □

Can the condition "s-" in Theorem 3.14 be omitted? We give a negative answer to this question.

EXAMPLE 3.15. – There exists a space X, which has a point-star network $\{\mathcal{P}_n\}$ consisting of cs-covers of X, but $\mathcal{P}_n$ is not a wsn-cover of X for every $n \in \mathbb{N}$.

PROOF. – Let X be the closed interval [0,1]. For $x \in X$ and $n \in \mathbb{N}$, we write $B_n(x) = \{y \in X : |y - x| < 1/n\}$, put $\mathcal{A}_{n,x} = \{S \cup \{x\} : S$ is a sequence converging to $x \in X$ and $S \subset B_n(x)\}$, and put $\mathcal{P}_n = \bigcup_{x \in X} \mathcal{A}_{n,x}$.

Claim 1: $\{\mathcal{P}_n\}$ is a point-star network of X.

Let $x \in U$ with U open in X. Then there exists $n \in \mathbb{N}$ such that $x \in B_n(x) \subset U$. Put $m = 2n$, then $st(x, \mathcal{P}_m) \subset U$. In fact, if $y \in st(x, \mathcal{P}_m)$, then there exists $z \in X$ such that $x, y \in S \cup \{z\} \in \mathcal{A}_{m,z}$, so $|y - z| < 1/m$ and $|x - z| < 1/m$. Thus $|x - y| < 2/m = 1/n$, so $y \in B_n(x) \subset U$. This proves that $st(x, \mathcal{P}_m) \subset U$, so $\{\mathcal{P}_n\}$ is a point-star network of X.

Claim 2: $\mathcal{P}_n$ is a *cs*-cover of X for every $n \in \mathbb{N}$.

Let $S = \{x_k\}$ be a sequence converging to x in X, S is eventually in $(S \cap B_n(x)) \cup \{x\}$. It is easy to see that $(S \cap B_n(x)) \cup \{x\} \in \mathcal{A}_{n,x} \subset \mathcal{P}_n$. So $\mathcal{P}_n$ is a *cs*-cover of X.

Claim 3: $\mathcal{P}_n$ is not a *wsn*-cover of X for every $n \in \mathbb{N}$.

It is clear.

Thus we complete the proof of this example. □

REMARK 3.16. – Let X and $\{\mathcal{P}_n\}$ be given as in Example 3.15. Then, for Ponomarev-system $(f, M, X, \{P_n\})$, f is sequence-covering from Theorem 3.7 and Claim 2 in Example 3.15 (note: f is also a π-mapping from Remark 3.3), and f is not 1-sequence-covering from Theorem 3.10 and Claim 3 in Example 3.15. So "s-" in Theorem 3.14 can not be omitted.

REMARK 3.17. – Every sequence-covering, compact mapping from a metric space is 1-sequence-covering. The following question was posed by S. Lin in [6, Question 3.4.3]: Is every sequence-covering, π-mapping from a metric space 1-sequence-covering? The answer is negative. In fact, let f be a mapping in Remark 3.16. Then f is a sequence-covering, π-mapping from a metric space M, but it is not 1-sequence-covering.

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Ying Ge: Department of Mathematics,
Suzhou University, Suzhou, 215006, P.R.China
e-mail: geying@pub.sz.jsinfo.net

Shou Lin: Department of Mathematics,
Ningde Teachers' College, Fujian 352100, P.R.China
e-mail: linshou@public.ndptt.fj.cn

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