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On countable-to-one maps ☆

Chuan Liu [a,*], Shou Lin [b]

[a] *Department of Mathematics, Ohio University-Zanesville, Ohio, OH 43701, USA*
[b] *Institute of Mathematics, Ningde Teachers' College, Fujian 352100, PR China*

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Abstract

In this paper, it is proved that a space with a point-countable base is an open, countable-to-one image of a metric space, and a quotient, countable-to-one image of a metric space is characterized by a point-countable $\aleph_0$-weak base. Examples are provided in order to answer negatively questions posed by Gruenhage et al. [G. Gruenhage, E. Michael, Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math. 113 (1984) 303–332] and Tanaka [Y. Tanaka, Closed maps and symmetric spaces, Questions Answers Gen. Topology 11 (1993) 215–233].

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1. Introduction

The certain images of metric spaces have been studied extensively in the past years [6]. It is well known that a T_0-space has a point-countable base if and only if it is an open s-image of a metric space [3], here $f: X \to Y$ is an s-map if each fiber $f^{-1}(y)$ is separable in X. G. Gruenhage et al. [4] showed that spaces determined by point-countable covers are preserved by quotient maps with countable fibers. Every countable-to-one map is an s-map. Are quotient countable-to-one images on metric spaces and quotient s-images on metric spaces coincident? The question is discussed and some related results are obtained in this paper.

Throughout this paper, all spaces are assumed to be T_2, all maps are continuous and onto. Denote real, irrational and rational numbers by $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$, respectively. We refer the reader to [2] for notations and terminology not explicitly given here.

2. Main results

Theorem 1. *The following are equivalent for a space X:*

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* Corresponding author. Tel.: +1 740 588 1419; fax: +1 740 453 6161.
E-mail addresses: liuc1@ohio.edu (C. Liu), linshou@public.ndptt.fj.cn (S. Lin).

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(1) *X has a point-countable base.*
(2) *X is an open s-image of a metric space.*
(3) *X is an open, countable-to-one image of a metric space.*

Proof. It is well known that (1) and (2) are equivalent. (3) ⇒ (2) is obvious. We prove that (2) ⇒ (3).

Let $f : M \to X$ be an open s-map from a metric space M onto the space X. For each $x \in X$, let D_x denote a countable dense subset of $f^{-1}(x)$ because $f^{-1}(x)$ is separable. Put $D = \bigcup\{D_x\colon\ x \in X\}$, and $g = f|_D : D \to X$. Then g is a countable-to-one map. We prove that g is open. Let U be an open subset of D. There is an open subset V in M such that $U = V \cap D$. If $g(U)$ is not open in X, there is $x \in g(U) \cap \overline{X \setminus g(U)}$. Since X is first countable, there is a sequence $\{x_n\}$ in $X \setminus g(U)$ with $x_n \to x$ in X. Because $x \in f(V)$ and $f(V)$ is open in X, without loss of generality, we can assume that each $x_n \in f(V)$. Thus $f^{-1}(x_n) \cap V \neq \emptyset$, and $D_{x_n} \cap V \neq \emptyset$. Pick $y_n \in D_{x_n} \cap V \subset U$, then $x_n = g(y_n) \in g(U)$, a contradiction. Thus $g(U)$ is open in X. Hence g is an open map and X is an open, countable-to-one image of the metric space D. □

Definition 2. Let $\mathcal{B}$ be a family of subsets of a space X. $\mathcal{B}$ is said to be an $\aleph_0$-*weak base* for X if $\mathcal{B} = \bigcup\{\mathcal{B}_x(n)\colon\ x \in X,\ n \in \mathbb{N}\}$ satisfies

(1) For each $x \in X$, $n \in \mathbb{N}$, $\mathcal{B}_x(n)$ is closed under finite intersections and $x \in \bigcap \mathcal{B}_x(n)$.
(2) A subset U of X is open if and only if whenever $x \in U$ and $n \in \mathbb{N}$, there exists a $B_x(n) \in \mathcal{B}_x(n)$ such that $B_x(n) \subset U$.

X is called $\aleph_0$-*weakly first-countable* [10] or *weakly quasi-first-countable* in the sense of Sirois-Dumais [9] if $\mathcal{B}_x(n)$ is countable for each $x \in X$, $n \in \mathbb{N}$.

If $\mathcal{B}_x(n) = \mathcal{B}_x(1)$ for each $n \in \mathbb{N}$ in the definition of $\aleph_0$-weak bases, the $\mathcal{B}$ is said to be a *weak base* for X [1]. X is called *weakly first-countable* or *g-first countable* in the sense of Arhangel'skiĭ [1] if $\mathcal{B}_x(1)$ is countable for each $x \in X$.

Let X be a space. $P \subset X$ is called a *sequential neighborhood* of x in X, if each sequence converging to x in X is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. X is called a *sequential space* if each sequential open subset of X is open.

Lemma 3. [9] *Every $\aleph_0$-weakly first-countable space is sequential.*

Let $f : X \to Y$ be a map. f is called *subsequence-covering* if whenever L is a convergent sequence in Y there is a convergent sequence S in X such that $f(S)$ is a subsequence of L.

Lemma 4. [6] *Let $f : X \to Y$ be a map, and Y a sequential space. Then f is quotient if and only if Y is a sequential space and f is subsequence-covering.*

Theorem 5. *X is a quotient, countable-to-one image of a metric space if and only if X has a point-countable $\aleph_0$-weak base.*

Proof. *Necessity*. Let $f : M \to X$ be a quotient, countable-to-one map from a metric space M onto the space X. Let $\mathcal{B}$ be a point-countable base for M. For each $y \in M$, let $\mathcal{B}_y \subset \mathcal{B}$ be a countable, decreasing local base at y in M. Put $\mathcal{B}' = \{\mathcal{B}_y\colon\ y \in M\}$. Then $\mathcal{B}'$ is a point-countable family of M. Since f is a countable-to-one map, $f(\mathcal{B}')$ is point-countable in X. We shall check that $f(\mathcal{B}')$ is an $\aleph_0$-weak base for X.

For each $y \in M$, denote $\mathcal{B}_y$ by $\{B_{y,i}\colon\ i \in \mathbb{N}\}$ with each $B_{y,i+1} \subset B_{y,i}$. For each $x \in X$, denote $f^{-1}(x)$ by $\{x_n\colon\ n \in \mathbb{N}\}$. Let $\mathcal{P}_x(n) = f(\mathcal{B}_{x_n})$. Then $f(\mathcal{B}') = \bigcup\{\mathcal{P}_x(n)\colon\ x \in X,\ n \in \mathbb{N}\}$. Let U be open in X. For each $x \in U$, $n \in \mathbb{N}$, $x_n \in f^{-1}(U)$, then $B_{x_n,i} \subset f^{-1}(U)$ for some $i \in \mathbb{N}$, thus $f(B_{x_n,i}) \in \mathcal{P}_x(n)$ and $f(B_{x_n,i}) \subset U$. On the other hand, let U be a subset of X satisfying for each $x \in U, n \in \mathbb{N}$, there exist $i \in \mathbb{N}$ such that $f(B_{x_n,i}) \subset U$. We prove that U is open in X. Since f is quotient, X is a sequential space by Lemma 4, it suffices to prove that U is sequential open in X. Suppose that U is not sequential open, there is a sequence L in $X \setminus U$ converging to $x \in U$. Since f is a quotient map, there is a sequence S converging to some $x_n \in f^{-1}(x)$ in M such that $f(S)$ is a subsequence of L by Lemma 4. Since the sequence S is eventually in $B_{x_n,i}$, thus the sequence $f(S)$ is eventually in $f(B_{x_n,i}) \subset U$, a contradiction. Thus U is sequential open. Hence, X has a point-countable $\aleph_0$-weak base.

Sufficiency. Let $\mathcal{B} = \bigcup\{\mathcal{B}_x(n)\colon x \in X, n \in \mathbb{N}\}$ be a point-countable $\aleph_0$-weak base, here each $\mathcal{B}_x(n) = \{B_x(n,m)\colon m \in \mathbb{N}\}$ with $B_x(n,m+1) \subset B_x(n,m)$ for each $m \in \mathbb{N}$. Then any subsequence $\mathcal{B}'_x$ of $\{B_x(n,m)\}_{m\in\mathbb{N}}$ is a network at x in X for each $x \in X$ and $n \in \mathbb{N}$, i.e., if U is an open neighborhood of x in X, then $x \in B \subset U$ for some $B \in \mathcal{B}'_x$. We rewrite $\mathcal{B} = \{B_\alpha\colon \alpha \in I\}$. Endow I with discrete topology and let I_i be a copy of I for each $i \in \mathbb{N}$. For convenience' sake, two families $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ of subsets of a space are said to be cofinal if there exist $n_0, m_0 \in \mathbb{N}$ such that $P_{n_0+i} = Q_{m_0+i}$ for every $i \in \mathbb{N}$. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} I_i\colon \{B_{\alpha_i}\}_{i\in\mathbb{N}} \text{ is cofinal to } \mathcal{B}_{x_\alpha}(n) \text{ for some } x_\alpha \in X,\ n \in \mathbb{N},\ \{B_{\alpha_i}\}_{i\in\mathbb{N}} \text{ is a network of } x_\alpha\right\}.$$

Define $f: M \to X$ as $f(\alpha) = x_\alpha$. It is easy to see that f is well-defined and onto because X is Hausdorff and each $\mathcal{B}_x(n)$ is a network of x in X for each $n \in \mathbb{N}$. And $f(\alpha) = \bigcap_{i\in\mathbb{N}} B_{\alpha_i}$ for each $\alpha = (\alpha_i) \in M$. Notice that $\mathcal{B}$ is point-countable, then f is countable-to-one. Also f is continuous, in fact, for any neighborhood U of x_α, since $\{B_{\alpha_i}\}_{i\in\mathbb{N}}$ is a network of x_α, there exists $m \in \mathbb{N}$ such that $B_{\alpha_m} \subset U$. Let $V = (I_1 \times \cdots \times \{\alpha_m\} \times I_{m+1} \times \cdots) \cap M$, then V is an open neighborhood of α in M and $f(V) \subset B_{\alpha_m} \subset U$, hence f is continuous.

To prove that f is a quotient map, we only need to prove that f is a subsequence-covering map by Lemmas 3 and 4.

Claim. *Let L be a sequence converging to $x \notin L$ in X. Then there exist a subsequence L' of L and $n_0 \in \mathbb{N}$ such that L' is eventually in $B_x(n_0,m)$ for any $m \in \mathbb{N}$.*

In fact, since the set L is not closed in X, there is $n_0 \in \mathbb{N}$ such that $B_x(n_0,m) \cap L \neq \emptyset$ for any $m \in \mathbb{N}$ by Definition 2. If $B_x(n_0,m) \cap L$ is finite for some $m \in \mathbb{N}$, then $B_x(n_0,k) \subset X \setminus (B_x(n_0,m) \cap L)$ for some $k \geqslant m$, thus $B_x(n_0,k) \cap L = \emptyset$, a contradiction. So $B_x(n_0,m) \cap L$ is infinite for any $m \in \mathbb{N}$, hence there exist a subsequence L' of L such that L' is eventually in $B_x(n_0,m)$ for any $m \in \mathbb{N}$. Denote L by $\{x_k\}$.

For each $i \in \mathbb{N}$, take $\alpha_i \in I_i$ with $B_{\alpha_i} = B_x(n_0,i)$. Let $\alpha = (\alpha_i)$, then $\alpha \in M$. For each $k \in \mathbb{N}$, put $n_k = \min\{m \in \mathbb{N}\colon x_k \notin B_x(n_0,m)\}$. Construct $z_k = (\beta_i(k)) \in \prod_{i\in\mathbb{N}} I_i$ as follows: if $i < n_k$, pick $\beta_i(k) \in I_i$ with $B_{\beta_i(k)} = B_x(n_0,i)$; otherwise pick $\beta_i(k) \in I_i$ such that $B_{\beta_i(k)} = B_{x_k}(1, i - n_k + 1)$. Then $\{B_{\beta_i}(k)\}_{i\in\mathbb{N}}$ is cofinal to $\mathcal{B}_{x_k}(1)$, thus $z_k \in M$ and $f(z_k) = x_k$. On the other hand, for each $i \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ such that $x_k \in B_x(n_0,i)$ for any $k \geqslant k_0$ because L' is eventually in $B_x(n_0,i)$. Then $i < n_k$ when $k \geqslant k_0$ by the definition of n_k, so $\beta_i(k) = \alpha_i$. It implies that the sequence $\{\beta_i(k)\}_{k\in\mathbb{N}}$ converges to α_i in the discrete space I_i. Hence, $\{z_k\}$ converges to α in M. Therefore, f is subsequence-covering, and f is a quotient map. □

It is natural to ask whether a quotient s-image of a metric space is a quotient, countable-to-one image of a metric space. The following example shows that the answer is "no".

Example 6. There is a closed image of a separable metric space, which is not $\aleph_0$-weakly first-countable.

Proof. Let $X = \mathbb{R}^2 \setminus (\mathbb{Q} \times \{0\})$ be endowed with the subspace topology of $\mathbb{R}^2$ with the usual topology. Then X is a separable metric space. Let Y be the quotient space from X by identifying $\mathbb{P} \times \{0\}$ to a point. It is obvious that the quotient map is a closed map. It has been proved that if an image of a metric space under a closed map is $\aleph_0$-weakly first-countable, then the each boundary of fibers is σ-compact by Theorem 2.1 in [7]. Since $\mathbb{P} \times \{0\}$ is not σ-compact in X, Y is not $\aleph_0$-weakly first-countable. □

We do not know if a quotient, σ-compact image of a metric space is a quotient, countable-to-one image of a metric space. We shall give a partial answer to the question.

Recall some related concepts. Let X be a space. A family $\mathcal{P}$ of subsets of X is said to be a *cs-network* [5] for X, if whenever U is an open set and a sequence $\{x_n\}$ in X converges to a point in U, then $\{x_n\}$ is eventually in P and $P \subset U$ for some $P \in \mathcal{P}$. A space is said to be an *$\aleph_0$-space* [5], if it has a countable *cs*-network.

Theorem 7. *The following are equivalent for a space X:*

(1) *X is a quotient, countable-to-one image of a separable metric space.*
(2) *X is a quotient, σ-compact image of a separable metric space.*
(3) *X is $\aleph_0$-weakly first-countable and a quotient image of a separable metric space.*
(4) *X has a countable $\aleph_0$-weak base.*
(5) *X is an $\aleph_0$-weakly first-countable and $\aleph_0$-space.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) due to [9]. (3) $\Rightarrow$ (5) is obvious [3]. We shall prove that (5) $\Rightarrow$ (4) $\Rightarrow$ (1). Let $\mathcal{P}$ be a countable cs-network which is closed under finite intersections. Let $\bigcup\{\mathcal{B}_x(n)\colon x \in X,\ n \in \mathbb{N}\}$ be an $\aleph_0$-weak base for X, here each $\mathcal{B}_x(n) = \{B_x(n,m)\colon m \in \mathbb{N}\}$ with $B_x(n,m+1) \subset B_x(n,m)$ for each $m \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{P}_x(n) = \{P \in \mathcal{P}\colon B_x(n,m) \subset P$ for some $m \in \mathbb{N}\}$. Then $\mathcal{P}_x(n)$ is closed under finite intersections.

$\mathcal{P}_x(n)$ is a network of x in X. In fact, suppose not, there is a neighborhood U of x in X such that $P \not\subset U$ for each $P \in \mathcal{P}_x(n)$. Put $\{P \in \mathcal{P}\colon x \in P \subset U\} = \{P_k\colon k \in \mathbb{N}\}$. Then $B_x(n,m) \not\subset P_k$ for any $m, k \in \mathbb{N}$. Pick $x_{mk} \in B_x(n,m) \setminus P_k$ for each $m \geqslant k$. Let $y_i = x_{mk}$, where $i = k + m(m-1)/2$. Then the sequence $\{y_i\}$ converges to x in X because $\{B_x(n,m)\}_{m\in\mathbb{N}}$ is a decreasing network of x in X. Since $\mathcal{P}$ is a cs-network for X, there exist $k, j \in \mathbb{N}$ such that $\{y_i\colon i \geqslant j\} \subset P_k$. Pick $i \geqslant j$ such that $y_i = x_{mk}$ for some $m \geqslant k$, then $x_{mk} \in P_k$, a contradiction.

Put $\mathcal{B} = \bigcup\{\mathcal{P}_x(n)\colon x \in X,\ n \in \mathbb{N}\}$. Then $\mathcal{B}$ is countable. We shall prove that $\mathcal{B}$ is an $\aleph_0$-weak base for X. We only need to prove that a subset V of X is open if whenever $x \in V$, $n \in \mathbb{N}$, there exists a $P_x(n) \in \mathcal{P}_x(n)$ such that $P_x(n) \subset V$. If V is not open in X, then V is not sequentially open because X is sequential by Lemma 3. There is a sequence L in $X \setminus V$ converging to a point $x \in V$. By the claim in the proof of Theorem 5, there exist a subsequence L' of L and $n_0 \in \mathbb{N}$ such that L' is eventually in $B_x(n_0, m)$ for any $m \in \mathbb{N}$. But $B_x(n_0, m) \subset P_x(n_0)$ for some $m \in \mathbb{N}$, so L' is eventually in $P_x(n_0) \subset V$, a contradiction. Hence, $\mathcal{B}$ is a countable $\aleph_0$-weak base for X.

(4) $\Rightarrow$ (1) similar to the proof of the Sufficiency of Theorem 5, where each I_i is countable and M is a separable metric space. $\square$

In the final part of this section we discuss the closed, countable-to-one images of metric spaces. A space X is said to be a *Fréchet space* if whenever $x \in \overline{A}$ in X there is a sequence in A which converges to x in X. A space X *is determined by* a cover $\mathcal{P}$ if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is open (closed) in P for each $P \in \mathcal{P}$ [4].

Theorem 8. *Let X be a Fréchet space determined by a countable cover of closed metric subsets. Then X is a closed, countable-to-one image of a metric space.*

Proof. Suppose that X is determined by a countable cover $\{X_n\}_{n\in\mathbb{N}}$ of closed metric subsets. Let $Y_n = X_n \setminus \bigcup\{X_i\colon i < n\}$, $Z_n = \overline{Y}_n$ for each $n \in \mathbb{N}$. Then $Y_i \cap Y_j = \emptyset$ if $i \neq j$. Note that if $x_n \in Y_n$, $\{x_n\colon n \in \mathbb{N}\}$ is a closed discrete subspace of X. In fact, if $A \subset \{x_n\colon n \in \mathbb{N}\}$, then $A \cap X_n \subset \{x_i\colon i \leqslant n\}$, which is closed in X_n for each $n \in \mathbb{N}$. Thus A is closed in X because X is determined by $\{X_n\colon n \in \mathbb{N}\}$.

Let $f : \bigoplus_{n\in\mathbb{N}} Z_n \to X$ be the obvious map. Then f is a countable-to-one map. Let A be a closed subset in $\bigoplus_{n\in\mathbb{N}} Z_n$.

Claim. *$f(A)$ is closed in X.*

Suppose not, there is a sequence $\{y_n\}$ in $f(A)$ with $y_n \to y \notin f(A)$. If $A \cap Z_{i_0} \cap \{y_n\colon n \in \mathbb{N}\}$ is infinite for some $i_0 \in \mathbb{N}$, $y \in A \cap Z_{i_0}$ as $A \cap Z_{i_0}$ is closed. Thus $y \in f(A)$, a contradiction. Hence, $A \cap Z_i \cap \{y_n\colon n \in \mathbb{N}\}$ is finite for each $i \in \mathbb{N}$. There is a subsequence $\{z_k\}$ of $\{y_n\}$ such that $z_k \in A \cap Z_{i_k}$ with each $i_k < i_{k+1}$. For each $k \in \mathbb{N}$, there is a sequence $\{x_n(k)\}$ in Y_{i_k} with $x_n(k) \to z_k$ in X. Thus $y \in \overline{\{x_n(k)\colon n, k \in \mathbb{N}\}}$. There is a sequence $\{x_{n_m}(k_m)\}_{m\in\mathbb{N}}$ converging to y, where each $k_m < k_{m+1}$. This is a contradiction because $\{x_{n_m}(k_m)\colon m \in \mathbb{N}\}$ is closed.

Hence, X is a closed, countable-to-one image of a metric space. $\square$

Example 9. There is a closed image of a countable metric space, which is not determined by a countable cover of metric subsets.

Proof. Let $X = \{(x, y)\colon x, y \in \mathbb{Q}\}$ be endowed with the subspace topology of $\mathbb{R}^2$ with the usual topology. Then X is a countable metric space. Let $A = \{(x, 0)\colon x \in \mathbb{Q}\}$. And let $Y = X/A$ be the quotient space from X by identifying all the points of A. Then Y is a closed image of a countable metric space. But Y is not determined by a countable cover of metric subsets by [12, Example 1.5(1)]. □

Question 10. How does one characterize, in intrinsic terms, closed, countable-to-one images of metric spaces?

3. Examples

In this section, two questions about open maps are negatively answered.

Question 11. [11] Does every open map preserve a weakly first-countable space?

We shall give an example which shows that an open, countable-to-one map may not preserve a weakly first-countable space.

Lemma 12. *Let $\mathbb{R}$ be the real numbers with the usual topology. Then $\mathbb{R}$ has ω_1 many disjoint dense subsets.*

Proof. For each $r \in \mathbb{R}$, put $r + \mathbb{Q} = \{r + q\colon q \in \mathbb{Q}\}$. Pick $p_1 \in \mathbb{P}$, then $p_1 + \mathbb{Q}$ is a dense subset that is disjoint with $\mathbb{Q}$. For $\alpha < \omega_1$, assume we have selected out disjoint dense subsets $\{p_\beta + \mathbb{Q}\colon \beta < \alpha\}$. Let $A = \mathbb{R} \setminus \bigcup\{p_\beta + \mathbb{Q}\colon \beta < \alpha\}$, pick $p_\alpha \in A \cap \mathbb{P}$, then $(p_\alpha + \mathbb{Q}) \cap (p_\beta + \mathbb{Q}) = \emptyset$ for each $\beta < \alpha$. Otherwise, there are $r_1, r_2 \in \mathbb{Q}$ such that $p_\alpha + r_1 = p_\beta + r_2$, so $p_\alpha = p_\beta + r_2 - r_1 \in p_\beta + \mathbb{Q}$, a contradiction. In this way, we obtain ω_1 many disjoint dense subsets $\{p_\alpha + \mathbb{Q}\colon \alpha < \omega_1\}$.

Let S_κ be the quotient space by identifying all limit points of the topological sum of κ many convergent sequences.

Example 13. There is an open map from a countable space with a countable weak base onto S_ω.

Proof. Let $R = \bigcup\{p_i + \mathbb{Q}\colon i \in \mathbb{N}\}$, where $\{p_i + \mathbb{Q}\colon i \in \mathbb{N}\}$ are disjoint dense subsets of $\mathbb{R}$ by Lemma 9. We write $p_i + \mathbb{Q} = \{p_i + r_n\colon n \in \mathbb{N}\}$. For each $p_i + r_n$, take a sequence $\{x_j(p_i, r_n)\}$ which converges to a point $x(p_i, r_n)$ in $\mathbb{R}^2$. Let M be the topological sum $R \oplus (\bigoplus\{\{x_j(p_i, r_n)\colon j \in \mathbb{N}\} \cup \{x(p_i, r_n)\}\colon i, n \in \mathbb{N}\})$. And let X be the quotient space of M by identifying $x(p_i, r_n)$ and $p_i + r_n$ to a point. Then X is a quotient, two-to-one image of the countable metric space M, hence X is a countable space with a countable weak base [8]. We write $S_\omega = \{\infty\} \cup \{z_j(i)\colon i, j \in \mathbb{N}\}$, where $z_j(i) \to \infty$ for each $i \in \mathbb{N}$. Define $f : X \to S_\omega$ as follows: $f(R) = \{\infty\}$, $f(x_j(p_i, r_n)) = z_j(i)$ for each $n \in \mathbb{N}$. It is not difficult to see that f is an open map.

Since S_ω is not weakly first-countable [8], it does not hold that spaces with weakly first-countability are preserved by open maps. □

Gruenhage et al. [4] proved that quotient s-images of metric spaces are preserved by quotient, countable-to-one maps; and pseudo-open, s-images of metric spaces are preserved by open, s-maps. They asked the following question in [4].

Question 14. Are quotient s-images of metric spaces preserved by open, s-maps?

We shall give a negative answer to this question by the following example, which also shows that an open compact map may not preserve a weakly first-countable space. This is another negative answer to Question 11.

Example 15. There is an open compact map from a quotient, two-to-one image of a metric space onto S_{ω_1}.

Proof. Let $\{p_\alpha + \mathbb{Q}\colon \alpha < \omega_1\}$ be disjoint families of dense subsets of $\mathbb{R}$ by Lemma 9. We write $\{x \in [0, 1]\colon x \in p_\alpha + \mathbb{Q}\} = \{p_\alpha + r_n\colon n \in \mathbb{N}\}$. For each $\alpha < \omega_1$ and $n, j \in \mathbb{N}$, let $x_j(p_\alpha, r_n) = (p_\alpha + r_n, 1/j)$ and $x(p_\alpha, r_n) = (p_\alpha + r_n, 0)$. Then $x_j(p_\alpha, r_n) \to x(p_\alpha, r_n)$ in $\mathbb{R}^2$. For $\alpha < \omega_1$, let $M_\alpha = (\bigcup_{n \in \mathbb{N}}\{x_j(p_\alpha, r_n)\colon j \in \mathbb{N}\} \cup \{x(p_\alpha, r_n)\}) \cup \{x_\alpha(j)\colon \alpha <$

$\omega_1,\ j \in \mathbb{N}\}$, here each $x_\alpha(j) \in \mathbb{R}^2$. Define a topology on M_α as follows: each $x_j(p_\alpha, r_n)$ is an isolated point; an element of a local base of $x_\alpha(j)$ in M_α has the form $\{x_\alpha(j)\} \cup \{x_j(p_\alpha, r_n)\colon\ n \geqslant m\}, \forall m \in \mathbb{N}$; an element of a local base of $x(p_\alpha, r_n)$ in M_α has the form $\{x(p_\alpha, r_n)\} \cup \{x_j(p_\alpha, r_n)\colon\ j \geqslant m\}$, $\forall m \in \mathbb{N}$. It is easy to see that M_α is a countable, regular and first-countable space, hence it is a metrizable space. Let M be the topological sum of $\{M_\alpha\colon\ \alpha < \omega_1\}$. Let X be the quotient space of a topological sum $[0,1] \oplus M$ by identifying $x(p_\alpha, r_n)$ and $p_\alpha + r_n$ to a point. Then X is a quotient, two-to-one image of a metric space. Thus X is also a weakly first-countable space [8].

We write $S_{\omega_1} = \{\infty\} \cup \{x_j(\alpha)\colon\ j \in \mathbb{N},\ \alpha < \omega_1\}$, where $x_j(\alpha) \to \infty$ for each $\alpha < \omega_1$. Define $f : X \to S_{\omega_1}$ by $f([0,1]) = \{\infty\}$, $f(\{x_j(p_\alpha, r_n)\colon\ n \in \mathbb{N}\} \cup \{x_\alpha(j)\}) = \{x_j(\alpha)\}$. It is easy to see that f is an open, compact, s-map.

Since S_{ω_1} is not any quotient, s-image of a metric space [6], it shows that an open, s-map may not preserve a quotient, s-image of a metric space. It is also proved that an open, compact map may not preserve a weakly first-countable space. □

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