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cs-REGULAR NETWORKS AND METRIZATION THEOREMS

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Abstract. The main purpose of this paper is to establish several metrization theorems related to cs-regular k-networks, which generalize some known results and answer a question of the second author in [6].

1. Introduction

In 1960, Arhangel'skiĭ [2] gave a metrization theorem, showing that a T_1-space is metrizable if and only if it has a regular base. In recent years, some topologists [5, 9] discussed the properties of various regular networks, such as LF-regular networks, PF-regular networks, and obtained relations between these networks and metrization. Jiang [4] introduced the concept of cs-regular collections as a generalization of regular collections. The second author in [6] proved that a T_2-space X is metrizable if and only if it has a cs-regular weak base, and posed the following question: " Is every first-countable and regular space with a cs-regular k-network metrizable?" In this paper, we study the properties of cs-regular k-networks, positively answer this question and show that every sequential space with cs-regular closed k-networks is metrizable, which generalizes some metrization theorems in [7] and [9].

Let X be a space, and $\{x_n\}$ a sequence converging to x in X, we denote $T(x) = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$, $T(x,m) = \{x\} \bigcup \{x_n : n \geq m\}$ for each $m \in \mathbb{N}$. Let us recall some definitions.

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Definition 1.1. Let X be a space and $\mathcal{P}$ a family of subsets of X.

(1) $\mathcal{P}$ is point-regular [1] if for each open set U in X, $\{P \in \mathcal{P} : P \not\subset U\}$ is point-finite at each point of U.

(2) $\mathcal{P}$ is cs-regular [4] if for each converging sequence $T(x)$ and an open neighborhood U of x in X, there exists $m \in \mathbb{N}$ such that $\{P \in \mathcal{P} : P \bigcap T(x,m) \neq \emptyset, P \not\subset U\}$ is finite.

(3) $\mathcal{P}$ is regular [2] if for each open set U in X, $\{P \in \mathcal{P} : P \not\subset U\}$ is locally finite at each point of U.

Aleksandrov [1] and Arhangel'skiĭ [2] studied the spaces with a point-regular base or a regular base, respectively. Junnila and Yajima [5] considered the concepts of point-regularity and regularity of networks, and the "point-regular" and "regular" are called "PF-regular" and "LF-regular", respectively. It is easy to show that a regular collection $\Longrightarrow$ cs-regular collection $\Longrightarrow$ point-regular collection.

Definition 1.2. Let (X, τ) be a space, and $\mathcal{P}$ a family of subsets of X.

(1) $\mathcal{P}$ is a k-network of X if for every compact subset K and $K \subset U \in \tau$, there exists a finite $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P}' \subset U$.

(2) $\mathcal{P}$ is a weak base of X if $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ satisfying that (a) For each $x \in X$, $x \in \bigcap \mathcal{P}_x$; (b) If $U, V \in \mathcal{P}_x$, then there exists $W \in \mathcal{P}_x$ such that $W \subset U \bigcap V$; (c) U is an open subset of X if and only if for every $x \in U$, there exists $P \in \mathcal{P}_x$ such that $P \subset U$. The $\mathcal{P}_x$ is called a weak base of x in X for each $x \in X$.

(3) $\mathcal{P}$ is a cs^*-network of X if for every converging sequence $T(x)$ and $x \in U \in \tau$, there exist a subsequence $T_1(x)$ of $T(x)$ and $P \in \mathcal{P}$ such that $T_1(x) \subset P \subset U$.

(4) $\mathcal{P}$ is a cs^*-cover of X if for every converging sequence $T(x)$, there exist a subsequence $T_1(x)$ of $T(x)$ and $P \in \mathcal{P}$ such that $T_1(x) \subset P$.

A closed k-network or a weak base is a cs^*-network.

Definition 1.3. Let X be a space.

(1) A sequence $\{\mathcal{P}_n\}$ of covers of X is a weak development of X if $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a weak base of x in X for each $x \in X$.

(2) A sequence $\{\mathcal{P}_n\}$ of covers of X is a point-star network of X if $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a network of x in X for each $x \in X$.

H. Martin proved the following result.

Theorem 1.4 [8]**.** *A T_2-space X is metrizable if and only if X has a weak development $\{\mathcal{P}_n\}$ such that $\{st^2(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a weak base of x for each $x \in X$.*

Throughout this paper, all spaces are assumed to be regular and T_1-spaces.

2. Metrization theorems

Lemma 2.1. *Let X be a sequential space, and $\mathcal{P}$ a point-regular cs^*-network which is closed under finite intersections. Then there exists a sequence $\{\mathcal{P}_n\}$ of cs^*-covers of X such that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ and $\{\mathcal{P}_n\}$ forms a weak development of X.*

Proof. Let $\mathcal{P}$ be a point-regular cs^*-network, then $\mathcal{P}$ has the following properties.

(1) If $x \in X$ and $\{P_n : n \in \mathbb{N}\}$ is an infinite subset of $(\mathcal{P})_x$, then $\{P_n : n \in \mathbb{N}\}$ is a network of x in X.

Obviously, every infinite subset of $(\mathcal{P})_x$ is a network of x in X.

(2) For each $P \in \mathcal{P}$, there exists $R \in \mathcal{P}$ such that $P \subset R$, and $Q \supset R$ if and only if $Q = R$ for any $Q \in \mathcal{P}$.

Suppose not, there exists $\{P_n\}$ satisfying that $P_1 = P$, $P_n \subset P_{n+1}$ and $P_n \neq P_{n+1}$. Let $\{x, y\} \subset P$ and $x \neq y$, then $\{P_n\}$ is a network of x, a contradiction.

Let $\mathcal{S}(X) = \{\{x\} : x \text{ is an isolated point in } X\}$, then $\mathcal{S}(X) \subset \mathcal{P}$.

Denote $\mathcal{P}^m = \{R \in \mathcal{P} : \text{if } R \subset P \in \mathcal{P}, \text{ then } R = P\}$, then $\mathcal{P}^m$ is a cs^*-cover of X from (2). Let $\mathcal{P}' = (\mathcal{P} \setminus P^m) \bigcup \mathcal{S}(X)$. We shall show that $\mathcal{P}'$ is also a point-regular cs^*-network.

It is obvious that $\mathcal{P}'$ is point-regular. Let $T(x)$ be a sequence converging to $x \in X$, and U an open neighborhood of x, then there exist $P_1 \in \mathcal{P}$ and a subsequence $T_1(x) \subset P_1 \subset U$. Pick $y \in T_1(x) \setminus \{x\}$, then $T_1(x) \setminus \{y\} \subset X \setminus \{y\}$, so there exist a subsequence $T_2(x)$ of $T_1(x)$ and $P_2 \in \mathcal{P}$ such that $T_2(x) \subset P_2 \subset X \setminus \{y\}$. Let $P = P_1 \bigcap P_2$, then $T_2(x) \subset P \subset U$, and $P \in \mathcal{P}'$, hence $\mathcal{P}'$ is a point-regular cs^*-network.

Let $\mathcal{P}_1 = \mathcal{P}^m, \mathcal{P}_{n+1} = [(\mathcal{P} \setminus \bigcup_{i=1}^n \mathcal{P}_i) \bigcup \mathcal{S}(X)]^m, n \in \mathbb{N}$, then $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ by (2) and each $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

For each $x \in X$ and $P_n \in (\mathcal{P}_n)_x$, $n \in \mathbb{N}$, if $x \in \mathcal{S}(X)$, then $P_m = \{x\}$ for some $m \in \mathbb{N}$, so $\{P_n : n \in \mathbb{N}\}$ is a network of x. If x is not an isolated point in X, then $\{P_n : n \in \mathbb{N}\}$ is an infinite subset of $(\mathcal{P})_x$, therefore, $\{P_n : n \in \mathbb{N}\}$ is a network of x by (1).

Hence, $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a network of x in X. On the other hand, since $\mathcal{P}_n$ is a cs^*-cover of X, $\mathrm{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x for each $x \in X$ and $n \in \mathbb{N}$. Since X is a sequential space, $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a weak base of x. Therefore $\{\mathcal{P}_n\}$ forms a weak development of X. □

Theorem 2.2. *The following are equivalent for a sequential space X:*

(1) X is metrizable.

(2) X has a cs-regular cs^*-network.

(3) X has a weak development $\{\mathcal{P}_n\}$ satisfying that for every converging sequence $T(x) \subset U \in \tau$, there exists $n \in \mathbb{N}$ such that $st(T(x), \mathcal{P}_n) \subset U$.

Proof. (1)⇒(2). Trivial.

(2)⇒ (3). Let $\mathcal{P}$ be a cs-regular cs^*-network, we can assume that $\mathcal{P}$ is closed under finite intersections, then $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ such that $\{\mathcal{P}_n\}$ forms a weak development of X by Lemma 2.1.

Let $T(x) \subset U \in \tau$. Since $\mathcal{P}$ is cs-regular, there is $m \in \mathbb{N}$ such that $\{P \in \mathcal{P} : P \bigcap T(x, m) \neq \emptyset, P \not\subset U\}$ is finite. So we can find $n_0 \in N$ satisfying that $\mathrm{st}(T(x, m), \mathcal{P}_n) \subset U$ whenever $n > n_0$.

For each $i < m$, pick $n_i \in \mathbb{N}$ such that $\mathrm{st}(x_i, \mathcal{P}_{n_i}) \subset U$. Let $k_0 = \max\{n_i : 0 \leq i < m\}$, then $\mathrm{st}(T(x), \mathcal{P}_n) \subset U$ whenever $n > k_0$.

(3)⇒(1). Let $\{\mathcal{P}_n\}$ be a weak development of X satisfying (3) and that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. By Theorem 1.4, we only need to show that for each $x \in X$, $\{\mathrm{st}^2(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ forms a weak base of x. Suppose not, then for some $x \in X$ and an open neighborhood V of x, $\mathrm{st}^2(x, \mathcal{P}_n) \not\subset V$ for each $n \in \mathbb{N}$, thus there exists $P_n \in \mathcal{P}_n$ such that $\mathrm{st}(x, \mathcal{P}_n) \bigcap P_n \neq \emptyset$ and $P_n \not\subset V$. Pick $x_n \in \mathrm{st}(x, \mathcal{P}_n) \bigcap P_n$, then the sequence $\{x_n\}$ converges to x, so $T(x, m_0) \subset V$ for some $m_0 \in \mathbb{N}$. By the property of $\{\mathcal{P}_n\}$, there exists $n_0 \in \mathbb{N}$ satisfying $\mathrm{st}(T(x, m_0), \mathcal{P}_{n_0}) \subset V$. Thus $P_{n_0+m_0} \subset \mathrm{st}(T(x, m_0), \mathcal{P}_{n_0+m_0}) \subset \mathrm{st}(T(x, m_0), \mathcal{P}_{n_0}) \subset V$, a contradiction , so X is metrizable. □

Corollary 2.3. *A sequential space with a cs-regular closed k-network is metrizable.*

Corollary 2.4 [7]**.** *A k-space with a regular k-network is metrizable.*

Proof. Let X be a k-space with a regular k-network. By the regularity of X and Lemma 1.1 in [5], X has a regular closed k-network. We know that this k-network is point-countable from the proof of Theorem 6 in [7], so X is a sequential space by Corollary 3.4 in [3], hence X is metrizable. □

The following result answers positively a question posed in [6].

Theorem 2.5. *Every first countable space with a cs-regular k-network is metrizable.*

Proof. Let $\mathcal{P}$ be a cs-regular k-network of a first countable space X, and $\overline{\mathcal{P}} = \{\overline{P} : P \in \mathcal{P}\}$. Then $\overline{\mathcal{P}}$ is a closed k-network of X. We shall show that $\overline{\mathcal{P}}$ is cs-regular.

First, $\overline{\mathcal{P}}$ is point-regular. Let $x \in U \in \tau, T(x) = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$. Without loss of generality, we can assume that $T(x) \subset U$. Pick $V \in \tau$ satisfying that $x \in V \subset \overline{V} \subset U$. If $\{\overline{P} : P \in \mathcal{P}, x \in \overline{P} \not\subset U\}$ is not a finite set, then there exists a sequence $\{P_n\}$ consisting of distinct elements of $\mathcal{P}$ such that $x \in \overline{P_n} \not\subset U$. Since X is a Fréchet space, there is a sequence $\{x_{nj}\}$ in P_n converging to x for each $n \in \mathbb{N}$. From the first countability of X, we can pick a sequence $\{x_{nj(n)}\}_{n\in\mathbb{N}}$ converging to x of the set $\{x_{nj} : n, j \in \mathbb{N}\}$ such that all $j(n)^{'}$s are distinct. Let $T_1(x) = \{x\} \bigcup \{x_{nj(n)} : n \in \mathbb{N}\}$. Since $\mathcal{P}$ is cs-regular, $\{P \in \mathcal{P} : T_1(x, m) \bigcap P \neq \emptyset, P \not\subset V\}$ is finite for some $m \in \mathbb{N}$. One the other hand, if $k \geq m$, then $P_k \bigcap T_1(x, m) \neq \emptyset$ and $P_k \not\subset V$, a contradiction, so $\overline{\mathcal{P}}$ is a point-regular k-network.

Let $\mathcal{F} = \{\overline{P} \not\subset U : P \in \mathcal{P}, T(x) \bigcap \overline{P} \neq \emptyset\}$, then $\mathcal{F}$ is finite. Suppose not, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{P_k\}$ consisting of distinct elements of $\mathcal{P}$ such that $x_{n_k} \in \overline{P_k} \not\subset U$. For each $k \in \mathbb{N}$, pick a sequence $\{y_{kj}\}$ in P_k converging to x_{n_k}. By the first countability of X, we can find a sequence $\{y_{kj(k)}\}_{k\in\mathbb{N}}$ converging to x of the set $\{y_{kj} : k, j \in \mathbb{N}\}$ such that all $j(k)^{'}$s are distinct, which contradicts the cs-regularity of $\mathcal{P}$ by repeating the process of the last paragraph, so $\overline{\mathcal{P}}$ is a cs-regular closed k-network of X. Therefore X is metrizable. □

3. Example

In this section, we shall give some examples to show the necessity of conditions in main theorems of this paper.

The following example explains that the condition of sequential spaces in Theorem 2.2 can't be weakened to one of k-spaces.

Example 3.1. A compact space with a cs-regular cs^*-network is not sequential.

Let X be the Čech-Stone compactification $\beta\mathbb{N}$ of $\mathbb{N}$, then X is a compact space without any non-trivial converging sequence. Obviously, X is not a sequential space, and $\{\{x\} : x \in X\}$ is a cs-regular cs^*-network of X. □

In Corollary 2.3, the closed property of k-networks is important. And in Theorem 2.5, the first countability can't be weakened to Fréchetness.

Example 3.2. Sequential fan S_ω is a Fréchet space with a cs-regular k-network, which is not first countable.

We only need to show that S_ω has a cs-regular k-network. Let S_ω=$\{x_0\} \bigcup \{x_{nm} : n, m \in \mathbb{N}\}$, where the sequence $\{x_{nm}\}_{m\in\mathbb{N}}$ converging to x_0 for each $n \in \mathbb{N}$. For every $n, m \in \mathbb{N}$, denote $V(n, m) = \{x_{nj} : j \geq m\}$, then collection $\{\{x\} : x \in S_\omega\} \bigcup \{V(n, m) : n, m \in \mathbb{N}\}$ is a cs-regular k-network. □

Finally, we show the importance of regularity of spaces in this paper.

Example 3.3. Half-disc topological space X is a first-countable, T_2-space with a regular k-network, but X is not a regular space.

Let τ be Euclidean topology of $\mathbb{R}^2$. $S = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, $L = \{(x, 0) : x \in \mathbb{R}\}$ and $X = S \bigcup L$. X is endowed the following topology $\tau^* = \tau_{|X} \bigcup \{\{x\} \cup (S \cap U) : x \in L, x \in U \in \tau\}$, then (X, τ^*) is called a half-disc topological space [10]. From the proof in [10], X is a first-countable, T_2 and non-regular space. Next, we show that X has a regular k-network.

For every $x \in \mathbb{R}^2$, $r > 0$, let $B(x, r)$ be the open ball in $(\mathbb{R}^2, \tau)$ with center x and radius r. For each $i \in \mathbb{N}$, let $\mathcal{B}_i$ be a locally finite open refinement of open cover $\{B(x, 1/4i) : x \in \mathbb{R}^2\}$ in $(\mathbb{R}^2, \tau)$, then $\mathcal{B} = \bigcup_{i=1}^\infty \mathcal{B}_i$ is a regular base in $(\mathbb{R}^2, \tau)$. In fact, for each $x \in \mathbb{R}^2$ and an open neighborhood O of x in $(\mathbb{R}^2, \tau)$, there exists $i \in \mathbb{N}$ such that $B(x, 1/i) \subset O$, let $V_0 = B(x, 1/2i)$, for every $k \leq i$, since $\mathcal{B}_k$ is locally finite, there exists an open neighborhood V_k such that V_k only meets finite many elements of $\mathcal{B}_k$. Let $V = \bigcap_{k=0}^i V_k$, then V is an open neighborhood of x and $\{B \in \mathcal{B} : B \bigcap V \neq \emptyset, B \not\subset O\}$ is finite. Put $\mathcal{P} = \{\{p\} : p \in L\} \bigcup \mathcal{B}_{|S}$, then $\mathcal{P}$ is a regular collection, so we only need to show it is a k-network of X. Let K be a non-empty compact subset of X, and U an open neighborhood of K in X. For each $x \in X$, let $\{P \in \mathcal{P} : x \in P \subset U\} = \{P_n(x) : n \in \mathbb{N}\}$, then there exists a finite subset of $\{P_n(x) : x \in K, n \in \mathbb{N}\}$ covering K. Suppose not, we can pick out a sequence $\{p_n\}$ in K such that $p_n \notin P_i(p_j)$ for each $i, j < n$. Since K is first-countable, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ converging to $p \in K$. By the discreteness of L, we can assume $p_{n_k} \in S$ for each $k \in \mathbb{N}$, then $\{p_{n_k}\}$ converging to p, also, $\mathcal{B}$ is a base of τ, so there are $B \in \mathcal{B}$ and $m \in \mathbb{N}$ such that $\{p_{n_k} : k \geq m\} \subset B \bigcap S \subset U$. Thus $B \bigcap S = P_i(p_j)$ for some $i, j \in \mathbb{N}$, hence there exists $n > i, j$ such that $p_n \in P_i(p_j)$, a contradiction. Therefore, $\mathcal{P}$ is a k-network of X, and X has a regular k-network.

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