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ALMOST-OPEN MAPS, SEQUENCE-COVERING MAPS AND SN-NETWORKS*

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In this paper, some relationships among the images of metric spaces under almost-open maps or sequence-covering maps and spaces determined by certain sn-networks are established, and some mapping theorems about generalized metric spaces are obtained.

Key Words: Sequentially quotient maps; sequence-covering maps; quotient maps; almost-open maps; weak bases; sn-networks.

To find the internal characterizations of certain images of metric spaces is one of the central questions in general topology. Since Arhangel'skiĭ [1] published the famous paper "Mappings and Spaces", the behavior of certain images on metric spaces has attracted considerable attention, and some noticeable results have been obtained. Weak bases introduced by Arhangel'skiĭ [1] have played an important role in the study of symmetric spaces and

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quotient images of metric spaces. But the spaces determined by certain weak bases is hard to discuss mapping properties because invariance of weak bases is bad at some maps. For example, a g-metrizable space (i.e., a regular space with a σ-locally finite weak base) is not preserved by perfect maps [14].

In recent years, sequence-covering maps introduced by Siwice [12] cause attention once again because it is closely related to the question about compact-covering and s-images of metric spaces [7, 8, 10]. It is discovered that sequence-covering maps is good for preservation of weak bases [9]. On the other hand, sn-networks determined by convergent sequences are more convenient than weak bases. In this paper, some relationships among the images of metric spaces under almost-open maps or sequence-covering maps and spaces determined by certain sn-networks are established. Throughout this paper, all spaces are T_2, all maps continuous and onto, and $\mathbb{N}$ is the set of positive integer numbers. Let us recall some definitions. Refer to [3] or [5] for terms which are not defined here.

Let $f : X \to Y$ be a map. f is *quotient* if whenever $f^{-1}(U)$ is open in X, then U is open in Y; f is *pseudo-open* if for each $y \in Y$ and a neighborhood U of $f^{-1}(y)$ in $X, f(U)$ is a neighborhood of y in Y; f is *sequence-covering* [12] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$; f is *sequentially quotient* [2] if whenever $\{y_n\}$ is a convergent sequence in Y there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$; f is *1- sequence-covering* [9] if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_k \in f^{-1}(y_n)$; f is 1 *-sequentially quotient* if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_k\}$ converging to x in X with each $x_k \in f^{-1}(y_{n_k})$.

The sequence-covering maps above-mentioned are different from the sequence-covering maps defined by Gruenhage, Michael and Tanaka [6].

It is obvious that

closed maps $\longrightarrow$ pseudo-open maps $\longrightarrow$ quotient maps;

and that

1-sequentially quotient maps

$\nearrow$ $\searrow$

1-sequence-covering maps sequentially quotient maps.

$\searrow$ $\nearrow$

sequence-covering maps

Let X be a space, and $P \subset X$. P is a *sequential neighborhood* of a point x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, i.e., $\{x_n\}$ is eventually in $P; P$ is a *sequentially open* subset of X if P is a sequential neighborhood of x in X for each $x \in P$. X is called a *sequential space* [4] if each sequentially open subset is open in X. X is called a *Fréchet space* [4] if $x \in \bar{P} \subset X$, there is a sequence in P converging to x in X.

It is easy to check that every Fréchet space is a sequential space, and for a space X, (1) $U \cap V$ is a sequential neighborhood of a point x in X if U, V are sequential neighborhoods of x in X; (2) A subset P of X is a sequential neighborhood of x if every sequence converging to x in X has a subsequence which is eventually P; (3) A subset P of a Fréchet space X is a neighborhood of x if P is a sequential neighborhood of x in X.

Lemma 1 — [2] Let $f : X \to Y$ be a map. Then

(1) If X is a sequential space, then f is a quotient map if and only if Y is a sequential space and f is a sequentially quotient map.

(2) If X is a Fréchet space, then f is a pseudo-open map if and only if Y is a Fréchet space and f is a sequentially quotient map. □

Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$,

(a) $\mathcal{P}_x$ is a network of x in X;

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a *weak base* [1] for X if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in $X; \mathcal{P}$ is called an *sn-network* [9] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a weak base(resp. an sn-network) defined the above-mentioned. Each $\mathcal{P}_x$ is called a local weak base(resp. a local sn-network) at x. A space X is called a g-first countable space(resp. an sn-first countable space) if X has a weak base(resp. an sn-network) such that the local weak base(resp. the local sn-network) of each point in X is a countable family.

First, we give some technical lemmas.

Lemma 2—Let $\mathcal{P}$ be a cover of a space X. Then

(1) If $\mathcal{P}$ is a weak base of $X, \mathcal{P}$ is an sn-network of X;

(2) If $\mathcal{P}$ is an sn-network of a sequential space $X, \mathcal{P}$ is a weak base of X. □

By the Lemma 2, we have the following relations.

First countable spaces $\longleftrightarrow$ g-first countable+Fréchet $\longleftrightarrow$ sn-first countable+Fréchet

$\downarrow$

g-first countable $\longleftrightarrow$ sn-first countable+sequential.

Lemma 3—Let $f : X \to Y$ be a map. Suppose $\{B_n\}_{n \in \mathbb{N}}$ is a decreasing network of a point x in X and each $f(B_n)$ is a sequential neighborhood of $f(x)$ in Y. If a sequence $\{y_n\}$ converges to $f(x)$ in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

PROOF: For each $n \in \mathbb{N}, f(B_n)$ is a sequential neighborhood of $f(x)$ in Y, there exists $i_n \in \mathbb{N}$ such that $y_i \in f(B_n)$ for each $i \geq i_n$, then $f^{-1}(y_i) \cap B_n \neq \emptyset$. We can assume that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, Put

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1 \\ f^{-1}(y_j) \cap B_n, & \text{if } i_n \leq j < i_{n+1}, n \in \mathbb{N}. \end{cases}$$

Then $x_j \in f^{-1}(y_j)$, and the sequence $\{x_j\}$ converges to x in X. □

***Theorem* 4** — Let $f : X \to Y$ be a 1-sequentially quotient map with X sn-first countable. Then Y is sn-first countable and f is 1-sequence-covering.

PROOF : Let $\mathcal{P}_x$ be a countable local sn-network at each point $x \in X$. Without losing generality, denote $\mathcal{P}_x$ by $\{P_{x,n}\}_{n \in \mathbb{N}}$ with every $P_{x,n+1} \subset P_{x,n}$. There is $x_y \in f^{-1}(y)$ satisfying the condition of 1-sequentially quotient maps for each $y \in Y$. Then each $f(P_{x_y,n})$ is a sequential neighborhood of y in Y. Let $\mathcal{Q}_y = \{f(P) : P \in \mathcal{P}_{x_y}\}$. Then $\bigcup_{y \in Y} \mathcal{Q}_y$ is an sn-network for Y, thus Y is sn-first countable. And f is 1-sequence-covering by Lemma 3. □

Corollary 5 — g-first countable spaces are preserved by 1-sequentially quotient and quotient maps.

PROOF : Let $f : X \to Y$ be a 1-sequentially quotient and quotient map. Then Y is an sn-first countable space and a sequential space by Theorem 4 and Lemma 1. thus Y is g-first countable by Lemma 2. □

The corollary is closely related to the following question posed by Y. Tanaka [14]: Is g-first countability preserved by open maps? Let $f : X \to Y$ be a map. f is called *almost-open* if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x in X then $f(U)$ is a neighborhood of $f(y)$ in Y. Every almost-open map is pseudo-open.

***Theorem* 6** — Let $f : X \to Y$ be a map.

(1) If X is first countable and f is almost-open, then f is 1-sequence-covering.

(2) If Y is Fréchet and f is 1-sequentially quotient, then f is almost-open.

PROOF : (1) For each $y \in Y$, there is $x_y \in f^{-1}(y)$ satisfying the condition of almost-open maps. Since X is a first-countable space, let $\{B_{y^n}\}_{n \in \mathbb{N}}$ be a decreasing local base of x_y in X. Then $\{f(B_{y,n})\}_{n \in \mathbb{N}}$ is a decreasing neighborhood base of y in Y. By Lemma 3, f is 1-sequence-covering.

(2) For each $y \in Y$, there is $x \in f^{-1}(y)$ satisfying the condition of 1-sequentially quotient maps. If U is a neighborhood of x in X, then $f(U)$ is a sequential neighborhood of y in y because f is 1-sequentially quotient. Thus $f(U)$ is a neighborhood of y in Y by Y is a Fréchet space, so f is almost-open.

Corollary 7 — Let $f : X \to Y$ be a map with X first-countable. Then f is almost-open if and only if f is 1-sequentially quotient and pseudo-open.

PROOF : It is easy to check by Theorem 6 and Lemma 1. □

We shall give some characterizations of metric spaces under sequence-covering maps. In 1960, V. Ponomarev proved that every first-countable space is an open image of some supspace of a Baire zero-dimensional metric spaces. Now, we generalize the Ponomarev's method as follows. Let $\mathcal{P}$ be a network of a space X. Denote $\mathcal{P}$ by $\{P_\alpha\}_{\alpha\in\Lambda}$. Λ is endowed with the discrete topology, and put $M = \{\alpha = (\alpha_i) \in \Lambda^\omega : \{\mathcal{P}_{\alpha_i}\}_{i\in\mathrm{N}}$ forms a network at some point x_α in $X\}$. Then M is a metric space. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Then $(f, M, X, \mathcal{P})$ is called a Ponomarev's system.

Lemma 8 — Let $(f, M, X, \mathcal{P})$ be a Ponomarev's system.

(1) f is a map if there exists a countable subset of $\mathcal{P}$ which forms a network at x for every $x \in X$.

(2) For every non-empty subset C of $X, f^{-1}(C)$ is a separable subspace of M if C only meets with countable many elements of $\mathcal{P}$.

(3) f is a 1-sequence-covering map if $\mathcal{P}$ is a countable local sn-network of X.

PROOF : (1) and (2) can be easily obtained by using the Ponomarev's method(see Theorem 6.1 in [6]). We only need show that (3) is hold. Let $\mathcal{P}$ be a countable local sn-network for X. For each $x \in X$, there is an sn-network $\{P_{\alpha_i}\}_{i\in\mathrm{N}}$ of x in X which is a countable subset of $\mathcal{P}$. Put $\beta = (\alpha_i) \in \Lambda^\omega$. Then $\beta \in f^{-1}(x)$. For each $n \in N$, let $B_n = \{(\gamma_i) \in M : \gamma_i = \alpha_i$ for each $i \leq n\}$. Then $\{B_n\}_{n\in\mathrm{N}}$ is a decreasing local base of β in M, and $f(B_n) = \cap_{i\leq n} P_{\alpha_i}$. In fact, suppose $\gamma = (\gamma_i) \in \bigcup_{i\in\mathrm{N}}$, then $f(\gamma) \in \bigcap_{i\in\mathrm{N}} P_{\gamma_i} \subset \bigcap_{i\leq n} P_{\alpha_i}$. Thus $f(B_n) \subset \bigcap_{i\leq n} P_{\alpha_i}$. On the other hand, let $z \in \cap_{i\leq n} P_{\alpha_i}$ take a network $\{P_{\delta_i}\}_{i\in\mathrm{N}}$ of z in X such that $\delta_i = \alpha_i$ when $i \leq n$. Let $\delta = (\delta_i) \in \Lambda^\omega$. Then $z = f(\delta) \in f(B_n)$, thus $\bigcap_{i\leq n} P_{\alpha_i} \subset f(B_n)$. Hence $f(B_n) = \bigcap_{i\leq n} P_{\alpha_i}$ is a sequential neighborhood of x in X. By Lemma 3, f is 1-sequence-covering. □

A map $f : X \to Y$ is said to be an s-map if each $f^{-1}(y)$ is separable.

***Theorem* 9** — The following are equivalent for a space X:

(1) X has a point-countable sn-network;

(2) X is a 1-sequence-covering and s-image of a metric space;

(3) X is a 1-sequentially quotient and s-image of a metric space.

PROOF : (1) $\Leftrightarrow$ (2) has been shown by the first author in this paper in [9] written in Chinese, here an outline of proof is given for reader convenience. (1) implies (2) by Lemma 8, and (2) implies (3) obviously. Let $f : M \to X$ be a 1-sequentially quotient and s-map here M is metric. Let $\mathfrak{B}$ be a σ-locally finite base of M by the Nagata-Smirnov's metrization theorem. For each $x \in X$, there is $a_x \in f^{-1}(x)$ satisfying the condition of 1-sequentially quotient maps. There is a countable subset $\mathfrak{B}_x$ of $\mathfrak{B}$ which is a local base of α_x in M. Then $f(\mathfrak{B}_x)$ is a countable local sn-network of x in X. Hence $\bigcup_{x\in X} f(\mathfrak{B}_x)$ is a point-countable sn-network for X. □

Corollary 10 — The following are equivalent for a space X

(1) X has a point-countable weak base;

(2) X is a 1-sequence-covering, quotient and s-image of a metric space,

(3) X is a 1-sequentially quotient, quotient and s-image of a metric space, □

Next, we shall discuss the π-images of metric spaces. For a metric space $(X, d), f : X \to Y$ is called a π-map if $d(f^{-1}(y), X - f^{-1}(V)) > 0$ for each $y \in Y$ and a neighborhood V of y in Y. Obviously, every compact map on metric spaces is a π-map. Let $\mathcal{P}$ be a cover of a space X. $\mathcal{P}$ is called an sn-cover for X if each element of $\mathcal{P}$ is a sequential neighborhood of some point in X, and for each $x \in X$ some $P \in \mathcal{P}$ is a sequential neighborhood of x. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X. $\{\mathcal{P}_n\}$ is called a point-star network [10] for X if $\{st(x, \mathcal{P}_n)\}_{n \in \mathbb{N}}$ is a network of x in X for each $x \in X$. "Point-star networks" were called "σ-strong networks" in [7], which is a generalization of the development of a space. A sequence $\{\mathcal{P}_n\}$ of covers of X is called a point-star network of sn-covers if $\{\mathcal{P}_n\}$ is a point-star network and each $\mathcal{P}_n$ is an sn-cover for X. It is easy to check that $\{\mathcal{P}_n\}$ is a point-star network if and only if $\{P_n\}_{n \in \mathbb{N}}$ is a network of x in X for each $x \in X$ with $x \in P_n \in \mathcal{P}_n$.

***Theorem* 11** — The following are equivalent for a space X:

(1) X is a 1-sequence-covering and π-image of a metric space;

(2) X is a 1-sequentially quotient and π-image of a metric space;

(3) X has a point-star network of sn-covers.

PROOF : (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3). Let (M, d) be a metric space and $f : M \to X$ be a 1-sequentially quotient and π-map. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{f(B(z, 1/n)) : z \in M\}$, here $B(z, 1/n) = \{y \in M : d(y, z) < 1/n\}$. Then $\{P_n\}$ is a point-star network for X. In fact, for each $x \in U$ with U open in X, there is $n \in \mathbb{N}$ such that $d(f^{-1}(x), Mf^{-1}(U)) > 1/n$. Take $m \in \mathbb{N}$ with $m \geq 2n$. If $x \in f(B(z, 1/m))$, then $f^{-1}(x) \cap B(z, 1/m) \neq \phi$. Suppose that $B(z, 1/m) \not\subset f^{-1}(U)$. Then $d(f^{-1}(x), M - f^{-1}(U)) < 2/m \leq 1/n$, a contradiction. Hence $B(z, 1/m) \subset f^{-1}(U)$, thus $f(B(z, 1/m)) \subset U$, so $st(x, \mathcal{P}_m) \subset U$. Therefore, $\{P_n\}$ is a point-star network for X. Since f is 1-sequentially quotient map, it is easy to check that each $\mathcal{P}_n$ is an sn-cover of X.

(3) $\Rightarrow$ (1). Let $\{\mathcal{P}_i\}$ be a point-star network of sn-covers for a space X. We extend the Ponomarev's system to the cover sequences of spaces as follows. For each $i \in \mathbb{N}$. put $\mathcal{P}_i = \{\mathcal{P}_{\alpha_i}\}_{\alpha \in \Lambda_i}$, and endow Λ_i with the discrete topology. Let $M = \{\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \{\mathcal{P}_{\alpha_i}\}_{i \in \mathbb{N}}$ forms a network at some point x_α in $X\}$. Then M is a metric space. Define $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. It is easy to check that f is a map. We shall show that f is a π-map. Let $p_k : \prod_{i \in \mathbb{N}} \Lambda_i \to \Lambda_k$ be the projective map for each $k \in \mathbb{N}$. For each $\alpha, \beta \in M$, define

$$d(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha = \beta, \\ \max\{1/k : p_k(\alpha) \neq p_k(\beta)\}, & \text{if } \alpha \neq \beta. \end{cases}$$

Then d is a metric of M. For each $x \in U$ with U open in X , there is $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. For each $\alpha \in f^{-1}(x)$, and $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $p_i(\alpha) = p_n(\beta)$ for each $i \leq n$, thus $x \in P_{p_n\alpha} = P_{p_n(\alpha)}$, so $f(\beta) \in \cap_{i \in \mathbb{N}} P_{p_n(\beta)} \subset U$, hence $d(f^{-1}(x), M -$

$f^{-1}(U)) \geq 1/n$. Therefore f is a π-map.

Now, we shall show that f is 1-sequence-covering. Let $x_0 \in X$. For each $i \in \mathbb{N}$, take $\alpha_i \in \Lambda_i$ such that P_{α_i} is a sequential neighborhood of x_0 in X. Then $\{P_{\alpha_i}\}_{i\in\mathbb{N}}$ is a network of x_0 in X. Put $\beta = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i$. Then $\beta \in f^{-1}(x_0)$. Let $\{x_n\}$ be a sequence converging to x_0 in X. For each $i, n \in \mathbb{N}$, take $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_{in}}$ with $\alpha_{in} = \alpha_i$ if $x_n \in P_{\alpha_i}$. Since the sequence $\{x_n\}$ is eventually in P_{α_i}, there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for each $n > n_i$, thus the sequence $\{\alpha_{in}\}$ converges to α_i in Λ_i. For each $n \in \mathbb{N}$, put $\beta_n = (\alpha_{in})$. Then $f(\beta_n) = \alpha_n$ and the sequence $\{\beta_n\}$ converges to β in M. Hence f is a 1-sequence-covering map. □

All spaces in the final are assume to be regular, and we shall further discuss some topological properties which is preserved by 1-sequentially quotient maps.

Let $\mathcal{P}$ be a collection of subsets of a space X. $\mathcal{P}$ is called a k-network [5] if for every compact subset K and a neighborhood V of K in X there exists a finite subset $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{F} \subset V$. A space with a σ-locally finite k-network is called an $\aleph$-space. A space with a σ-locally finite weak base is called a g-metrizable space [13]. S_{ω_1} is the quotient space obtained from the topological sum of ω_1 many non-trivial convergent sequences by identifying all the limit points to a single point. It is known that metric spaces $\longleftrightarrow$ g-metrizable spaces+Fréchet spaces $\rightarrow$ N-spaces+first countable spaces $\longleftrightarrow$ g-metrizable spaces $\rightarrow$ $\aleph$-spaces+g-first countable spaces $\rightarrow$ $\aleph$-spaces $\longleftrightarrow$ spaces which has a σ-hereditarily closure-preserving k-network and has not any (closed) copy of $S_{\omega_1 5}$.

***Theorem* 12** — $\aleph$-spaces are preserved by 1-sequentially quotient and closed maps.

PROOF : Let f : $X \rightarrow Y$ be a 1-sequentially quotient and closed map, here X is an N-space. Since f is closed, Y has a σ-hereditarily closure-preserving k-network. If Y is not an $\aleph$-space, Y has a closed subspace T which is homeomorphic to S_{ω_1}. Let $T = \{t\} \cup (\bigcup_{\alpha<\omega_1} T_\alpha)$, here the family $\{T_\alpha\}_{\alpha<\omega_1}$ is disjoint and each T_α is a sequence converging to t. Since f is 1-sequentially quotient, there is $s \in f^{-1}(t)$ such that whenever $\{y_n\}$ is a sequence converging to t in Y there is a sequence $\{x_k\}$ converging to s in X with each $x_k \in f^{-1}(y_{n_k})$. For each $\alpha < \omega_1$ there is a sequence X_α converging to s such that $f(X_\alpha)$ is a subsequence of the sequence T_α. Let $S = \{s\} \cup (\bigcup_{\alpha<\omega_1} X_\alpha)$,. Then S is homeomorphic to S_{ω_1}. In fact, it is obvious that the family $\{X_\alpha\}_{\alpha<\omega_1}$ is disjoint. Since each point of T_α is isolated in T, each point of X_α is isolated in S. For each $\alpha < \omega_1$ and each finite subset F_α of X_α, then $\{f(F_\alpha)\}_{\alpha<\omega_1}$ is discrete in Y. By the continuity of $f, \{F_\alpha\}_{\alpha<\omega_1}$ is discrete in X. Thus the subspace S of X is homeomorphic to S_{ω_1}, a contradiction. Therefore, Y is an $\aleph$-space. □

Corollary 13 — g-metrizability or metrizability is preserved by 1-sequentially quotient and closed maps.

PROOF : Since every g-metrizable space is equivalent to an $\aleph$-space with countable local weak bases, g-metrizability is preserved by 1-sequentially quotient and closed maps by Corollary 5 and Theorem 12. Since every metric space is equivalent to a g-metrizable, Fréchet space, metrizability is preserved by 1-sequentially quotient and closed maps by Corollary 7 and Lemma 1. □

S_ω is the quotient space obtained from the topological sum of ω many non-trivial convergent sequences by identifying all the limit points to a single point.

Example 14 — (1) S_ω is a sequence-covering, pseudo-open and s-image of a metric space [10]. Since S_ω is not g-first countable, it is not any 1-sequentially quotient image of a metric space by Lemma 1 and Corollary 5.

(2) There is a two-to-one quotient map $f : M \to Y$, with M a locally compact metric space, and Y completely regular, not point-countable sn-network [6]. Y is not any 1-sequentially quotient and s-image of a metric space by Theorem 9, thus f is not a 1-sequentially quotient map.

(3) There is a 1-sequence-covering and perfect map $f : X \to Y$ such that f is not almost-open. Let X_1 be the set $\omega_1 + 1$ of order numbers. X_1 is endowed with the following topology: Declaring each point $x \in X_1 - \{\omega_1\}$ isolated and the point ω_1 having a local base in $\omega_1 + 1$ with the usual order topology. Take $X_2 = X_1$, and let $X = X_1 \oplus X_2$. Then X is regular. The limit point in X_1 or X_2 is denoted by a_1 or a_2, respectively. Put $A = \{a_1, a_2\}$. Let Y be the quotient space X/A, and $q : X \to Y$ the quotient map. Then q is a perfect map (i.e., a closed and compact map), q is 1-sequence-covering because each convergent sequence in Y is trivial. Since $q(X_1)$ and $q(X_2)$ are not a neighborhood of $q(A)$ in Y, it is easy check that q is not almost-open.

(4) There is a perfect map $f : X \to Y$ such that f is not sequentially quotient.

Let X be the Stone-Čech compactification $\beta\mathbb{N}, Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the subspace topology of real line $\mathbb{R}$. Define $f : X \to Y$ by $f(\beta\mathbb{N} \setminus \mathbb{N}) = \{0\}$ and $f(n) = 1/n$ for each $n \in \mathbb{N}$. Then f is a non-sequentially quotient, perfect map.

(5) There is an open map $f : X \to Y$ such that f is not sequentially quotient.

Let $Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ with the subspace topology of real line $\mathbb{R}$, and $X = \{0\} \cup \mathbb{N}^2$. For each $n, i \in \mathbb{N}$, put $V(n,i) = \{(n,k) \in \mathbb{N}^2 : k \geq i\}$. X is endowed with the following topology: Each point in $\mathbb{N}^2$ is isolated; an element of a neighborhood base of 0 in X is $\{0\} \cup ((\bigcup_{n \geq m} V(n, i_n))$, here $m, i_n \in \mathbb{N}$. Define $f : X \to Y$ by $f(0) = 0$ and $f(n,i) = 1/n$ for each $n, i \in \mathbb{N}^2$. Then f is a non-sequentially quotient, open map. □

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