

A note on D-spaces

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Abstract. Every semi-stratifiable space or strong Σ -space has a σ -cushioned $(\text{mod}k)$ -network. In this paper it is showed that every space with a σ -cushioned $(\text{mod}k)$ -network is a D-space, which is a common generalization of some results about D-spaces.

Keywords: D-spaces, $(\text{mod}k)$ -networks, cushioned collections, semi-stratifiable spaces

Classification: 54E18, 54D20

A *neighborhood assignment* for a topological space (X, τ) is a function $\phi : X \rightarrow \tau$ such that $x \in \phi(x)$ for each $x \in X$. A space X is said to be a D-space if, for each neighborhood assignment ϕ for X , there exists a closed discrete subset D of X such that $\{\phi(d) : d \in D\}$ covers X . The first published results on D-spaces appear in [5], where it is proved that finite products of Sorgenfrey lines are D-spaces. Several interesting questions on D-spaces were raised by E.K. van Douwen and W.F. Pfeffer in [6]. It is still an open problem whether every regular Lindelöf space is a D-space. It is also asked whether there exists a subparacompact or metacompact space which is not a D-space. These questions are still open.

In [1] the authors study D-property in classes of generalized metric spaces. It is known that semi-stratifiable spaces are D-spaces [2], and strong Σ -spaces are D-spaces [3]. Semi-stratifiable spaces need not be strong Σ -spaces, and strong Σ -spaces need not be semi-stratifiable spaces [8]. It is therefore natural to look for a “common denominator” to the results mentioned above.

Throughout this paper, all spaces are assumed to be T_1 . We refer the reader to [7] for notations and terminology not explicitly given here.

Definition 1. A space X is said to be a *semi-stratifiable space* [4] if, for each open set U of X , one can assign a sequence $\{F(n, U)\}_{n \in \mathbb{N}}$ of closed subsets of X such that

- (1) $U = \bigcup_{n \in \mathbb{N}} F(n, U)$;
- (2) $F(n, U) \subset F(n, V)$ whenever $U \subset V$.

A correspondence $U \rightarrow \{F(n, U)\}_{n \in \mathbb{N}}$ is a *semi-stratification* for X whenever it satisfies the conditions (1) and (2).

This project was supported by NNSF of China (No.10571151).

A collection \mathcal{P} of pairs of subsets of a space X is called a *pair-network* for X if whenever $x \in U$ with U open in X , $x \in P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$. A collection \mathcal{P} of pairs of subsets of a space X is called *cushioned* if $\overline{\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}} \subset \bigcup\{P_2 : (P_1, P_2) \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$.

Lemma 2. *A space is a semi-stratifiable space if and only if it is a space with a σ -cushioned network.*

PROOF: Let (X, τ) be a semi-stratifiable space, and $U \rightarrow \{F(n, U)\}_{n \in \mathbb{N}}$ be a semi-stratification for X . For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{(F(n, U), U) : U \in \tau\}$. If $\tau' \subset \tau$, then $\overline{\bigcup_{U \in \tau'} F(n, U)} \subset F(n, \bigcup \tau') \subset \bigcup \tau'$. Thus \mathcal{P}_n is a cushioned collection of X . Hence, $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ is a σ -cushioned network for X .

Conversely, suppose that a space X has a σ -cushioned network $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$, where each \mathcal{P}_n is cushioned in X . For each $n \in \mathbb{N}$, $U \in \tau$, put $F(n, U) = \overline{\bigcup\{P_1 : (P_1, P_2) \in \bigcup_{i \leq n} \mathcal{P}_i, P_2 \subset U\}}$. It is obvious that $F(n, U) \subset U$ and $F(n, U) \subset F(n, V)$ whenever $U \subset V$. Let $x \in U \in \tau$. There exist $n \in \mathbb{N}$ and $(P_1, P_2) \in \mathcal{P}_n$ such that $x \in P_1 \subset P_2 \subset U$, thus $x \in F(n, U)$. Hence $U = \bigcup_{n \in \mathbb{N}} F(n, U)$. Therefore, $U \rightarrow \{F(n, U)\}_{n \in \mathbb{N}}$ is a semi-stratification for X . □

Definition 3. A collection \mathcal{P} of pairs of subsets of a space X is called a *(modk)-network* for X if, there is a cover \mathcal{K} of compact subsets of X such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X , then $K \subset P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$.

Theorem 4. *Spaces with a σ -cushioned (modk)-network are D-spaces.*

PROOF: Suppose a space X has a σ -cushioned (modk)-network $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ with respect to a cover \mathcal{K} of compact subsets, where each \mathcal{P}_n is cushioned and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$. Assume each \mathcal{P}_n is well-ordered.

Let ϕ be an arbitrary neighborhood assignment for X . First, a subset D_n of X is defined inductively as follows for each $n \in \mathbb{N}$.

Set $D_0 = \emptyset$. Assume $D_m \subset X$ is defined for all $0 < m < n$.

A finite subset D_α^n of X is defined inductively as follows for order numbers α .

Set $D_0^n = \emptyset$. Assume D_β^n is defined for all $0 < \beta < \alpha$. Put

$$U = \bigcup \left\{ \phi(d) : d \in \left(\bigcup \{D_\beta^n : \beta < \alpha\} \right) \cup D_{n-1} \right\}.$$

Denote R_α^n by the following requirement: There exist $K \in \mathcal{K}$, $(P_1, P_2) \in \mathcal{P}_n$ and $\{x_1, x_2, \dots, x_k\} \subset K \setminus U$ such that

$$K \subset P_1 \subset P_2 \subset U \cup \phi(x_1) \cup \phi(x_2) \cup \dots \cup \phi(x_k).$$

If R_α^n does not hold, the induction on α stops. Otherwise, let (P_1, P_2) be the first pair in \mathcal{P}_n that satisfies R_α^n , and put $D_\alpha^n = \{x_1, x_2, \dots, x_k\}$.

Let $D_n = (\bigcup_{\alpha < \gamma_n} D_\alpha^n) \cup D_{n-1}$ for some order number γ_n , where γ_n is the first ordinal number for which R_α^n does not hold.

Secondly, put $D = \bigcup \{D_n : n \in \mathbb{N}\}$, and we shall prove that X is a D-space by D .

Claim 1. $X = \bigcup_{d \in D} \phi(d)$.

If not, then $K \setminus \bigcup_{d \in D} \phi(d) \neq \emptyset$ for some $K \in \mathcal{K}$. Put $L = K \setminus \bigcup_{d \in D} \phi(d)$. L is a non-empty, compact subset of X . There is a finite subset $\{x_1, x_2, \dots, x_k\}$ of L such that $L \subset \phi(x_1) \cup \phi(x_2) \cup \dots \cup \phi(x_k)$ because ϕ is a neighborhood assignment for X . Denote $M = K \setminus (\phi(x_1) \cup \phi(x_2) \cup \dots \cup \phi(x_k))$. Then $M \subset K \setminus L \subset \bigcup_{d \in D} \phi(d)$. By the compactness of M , $M \subset \bigcup_{d \in D_j} \phi(d)$ for some $j \in \mathbb{N}$, thus $K \subset \phi(x_1) \cup \phi(x_2) \cup \dots \cup \phi(x_k) \cup (\bigcup_{d \in D_j} \phi(d))$. Since \mathcal{P} is a (mod k)-network for X with respect to \mathcal{K} , there are $m \in \mathbb{N}$ and $(P_1, P_2) \in \mathcal{P}_m$ such that

$$K \subset P_1 \subset P_2 \subset \phi(x_1) \cup \phi(x_2) \cup \dots \cup \phi(x_k) \cup \left(\bigcup_{d \in D_j} \phi(d) \right).$$

Put $n = \max\{j + 1, m\}$, $U = \bigcup \{\phi(d) : d \in D_{n-1}\}$. Then $\bigcup_{d \in D_j} \phi(d) \subset U$, $(P_1, P_2) \in \mathcal{P}_n$ and $\{x_1, x_2, \dots, x_k\} \subset K \setminus U$. Hence R_1^n holds, thus $D_1^n = \{x_1, x_2, \dots, x_k\}$, and $K \subset P_1 \subset P_2 \subset \bigcup_{d \in D} \phi(d)$, a contradiction.

Claim 2. Each D_n is a closed discrete subset of X .

D_0 is closed discrete in X . Suppose that D_{n-1} is closed discrete in X . It suffices to prove that $\bigcup_{\alpha < \gamma_n} D_\alpha^n$ is closed discrete in X in order to show that D_n is closed discrete in X . For each $\alpha < \gamma_n$, let $(P_{1\alpha}^n, P_{2\alpha}^n)$ be the first element in \mathcal{P}_n satisfying R_α^n . Assume that $x \in \overline{\bigcup_{\alpha < \gamma_n} D_\alpha^n}$. Since $\bigcup_{\alpha < \gamma_n} D_\alpha^n \subset \bigcup_{\alpha < \gamma_n} P_{1\alpha}^n$, then $x \in \bigcup_{\alpha < \gamma_n} P_{2\alpha}^n \subset \bigcup \{\phi(d) : d \in (\bigcup \{D_\alpha^n : \alpha < \gamma_n\}) \cup D_{n-1}\}$. And since each $D_\alpha^n \cap \bigcup_{d \in D_{n-1}} \phi(d) = \emptyset$, then $x \in \bigcup \{\phi(d) : d \in \bigcup_{\alpha < \gamma_n} D_\alpha^n\}$. Let α_x be the minimal element α in γ_n satisfying $x \in \bigcup \{\phi(d) : d \in D_\alpha^n\}$. Put $V = \bigcup \{\phi(d) : d \in D_{\alpha_x}^n\} \setminus \overline{\bigcup_{\alpha < \alpha_x} P_{1\alpha}^n}$. Then V is an open neighborhood of x in X , and $V \cap (\bigcup_{\alpha < \gamma_n} D_\alpha^n) \subset D_{\alpha_x}^n$ is finite. Hence, $\bigcup_{\alpha < \gamma_n} D_\alpha^n$ is closed discrete in X .

Claim 3. D is closed discrete in X .

For each $x \in X$, there is $n \in \mathbb{N}$ such that $x \in \bigcup_{d \in D_n} \phi(d)$ by Claim 1. Since D_n is closed discrete in X , there is an open neighborhood W of x in X such that $W \subset \bigcup_{d \in D_n} \phi(d)$ and W contains at most a point in D_n . Thus $W \cap D \subset ((\bigcup_{d \in D_n} \phi(d)) \cap (D \setminus D_n)) \cup (W \cap D_n)$. For each $y \in D \setminus D_n$, $y \in D_\alpha^m$ for some $m > n$ and some $\alpha < \gamma_m$. Put $U = \bigcup \{\phi(d) : d \in (\bigcup \{D_\beta^m : \beta < \alpha\}) \cup D_{m-1}\}$. Then $\bigcup_{d \in D_n} \phi(d) \subset U$, and $D_\alpha^m \cap U = \emptyset$, thus $y \notin \bigcup_{d \in D_n} \phi(d)$. Hence, $(D \setminus D_n) \cap (\bigcup_{d \in D_n} \phi(d)) = \emptyset$. This shows that W contains at most a point in D . Therefore, D is closed discrete in X . \square

A collection \mathcal{F} of closed subsets of a space X is called *closure-preserved* if $\bigcup \mathcal{F}'$ is closed for each $\mathcal{F}' \subset \mathcal{F}$. A space is called a *strong Σ -space* (*strong $\Sigma^\#$ -space*) [8] if there exists a σ -locally finite (σ -closure-preserved) collection \mathcal{F} of closed subsets, and a cover \mathcal{K} of X by compact subsets, such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X , then $K \subset F \subset U$ for some $F \in \mathcal{F}$. Every strong Σ -space is a strong $\Sigma^\#$ -space. If a collection \mathcal{F} of closed subsets of a space X is closure-preserved, then a collection $\{(F, F) : F \in \mathcal{F}\}$ of pairs of subsets of X is cushioned. Hence every strong $\Sigma^\#$ -space or semi-stratifiable space is a space with a σ -cushioned (mod k)-network. Thus Theorem 4 is a common generalization of Buzyakova's result [3] and Borges-Wehrly's result [2]. The following corollary is an unpublished result obtained by Liang-xue Peng.

Corollary 5. *Every strong $\Sigma^\#$ -space is a D-space.*

Acknowledgment. The author would like to thank Liang-xue Peng for useful information about D-spaces.

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(Received October 10, 2005, revised January 13, 2006)