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COVERING PROPERTIES OF k-SEMISTRATIFIABLE SPACES

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Abstract. k-semistratifiable spaces as a generalization of stratifiable spaces and $\aleph$-spaces have many important properties. In this paper, covering properties of k-semistratifiable spaces are discussed, and the following results are obtained: (1) every k-semistratifiable k-space is a hereditarily meta-Lindelöf space; (2) every k-semistratifiable, normal k-space is a hereditarily paracompact space.

Metric spaces have many good covering properties. Generalized metric spaces also have some similar covering properties. For example, M_1-spaces are paracompact spaces and σ-spaces are subparacompact spaces. Frank Siwiec [17] posed the following questions:

(S1) Are g-metrizable spaces normal spaces?

(S2) Are normal g-metrizable spaces paracompact spaces?

(S3) Are separable g-metrizable spaces the spaces with a countable weak base?

Can (S3) be changed to ask the following question?

(S4) Are g-metrizable spaces meta-Lindelöf spaces?

N. N. Jakovlev [9] announced the positive answers of questions (S2), (S3), and (S4). L. Foged [3], [5] discussed some equivalent conditions of g-metrizable spaces, established normality and covering properties in k- and $\aleph$-spaces, and answered all of Siwiec's questions. He proved

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(F1) there is a non-normal g-metrizable space;

(F2) under (MA+¬CH), there is a regular, non-monotonically normal space with a countable weak base;

(F3) every normal, k-space with a σ-locally finite k-network is a paracompact space;

(F4) every regular, k-space with a σ-locally finite k-network is a hereditarily meta-Lindelöf space.

Chuan Liu [12] and Liang-Xue Peng [16] proved that a result similar to (F3) and (F4), respectively, held for regular spaces with a σ-hereditarily closure-preserving k-network. Do the results hold for regular spaces with a σ-closure-preserving k-network? The regular spaces with a σ-closure-preserving k-network are k-semistratifiable spaces. In this paper, we shall further show that results similar to (F3) and (F4) hold for k-semistratifiable spaces. By a space we mean a *Hausdorff* topological space. Recalled below are some related concepts. Refer to [1] or [8] for terms which are not defined here.

Definition 1. Let X be a space.

(1) For $F \subset P \subset X$, P is said to be a *sequential neighborhood* of F in X if every sequence converging to a point of F is eventually in P.

(2) X is said to be a *sequential space* [6] if whenever a subset A of X is a sequential neighborhood of A, then A is open in X.

(3) X is said to be a k-space if whenever $K \cap A$ is closed in K for each compact subset K of X, then A is closed in X.

Definition 2 ([13])**.** A space X is said to be k-semistratifiable if for each open subset U of X there is a sequence $\{F(n,U)\}_{n\in\mathbb{N}}$ of closed subsets of X such that

(1) $U = \bigcup_{n\in\mathbb{N}} F(n,U)$;

(2) if $V \subset U$, then $F(n,V) \subset F(n,U)$;

(3) if a compact subset $K \subset U$, then $K \subset F(m,U)$ for some $m \in \mathbb{N}$.

The correspondence $U \to \{F(n,U)\}_{n\in\mathbb{N}}$ is said to be a *k-semistratification* for the space X.

Let $\mathcal{P}$ be a family of subsets of a space X. $\mathcal{P}$ is said to be a *discrete family* of X if there is an open neighborhood U of x in X such that U meets at most some element of $\mathcal{P}$ for each $x \in X$. $\mathcal{P}$ is said to be a *closure-preserving family* of X if $\overline{\cup\mathcal{P}'} = \cup\{\overline{P} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$. Obviously, a discrete family of a space X is closure-preserving. It is easy to check in [8] or [18]:

Metric spaces$\Longrightarrow$ g-metrizable spaces$\Longrightarrow$sequential spaces$\Longrightarrow$ k-spaces

$\Downarrow$ $\Downarrow$

M_1-spaces $\aleph$-spaces

$\Downarrow$ $\Downarrow$

stratifiable spaces$\Longrightarrow$regular spaces with a σ-closure-preserving k-network

$\Downarrow$

k-semistratifiable spaces$\Longrightarrow$subparacompact spaces.

Theorem 1. *Every k-semistratifiable k-space is a hereditarily meta-Lindelöf space.*

Proof: Suppose that X is a k-semistratifiable k-space and $U \to \{F(n,U)\}_{n\in\mathbb{N}}$ a k-semistratification for the space X. We can assume that each $F(n,U) \subset F(n+1,U)$. To complete the proof, it suffices to show that every open subspace of X is a meta-Lindelöf space [1]. Since the property of k-semistratifiable k-spaces is open hereditary [2], [13], we prove only that X is a meta-Lindelöf space.

(1) Every discrete family of closed subsets may be expanded to a point-countable family of open subsets of X.

Let $\mathcal{F}$ be a family of closed subsets of X. Put $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, put

$P^*_\alpha(\emptyset) = F_\alpha$, $\quad P_\alpha(\emptyset) = P^*_\alpha(\emptyset) \setminus \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\emptyset)}$, and

$\mathcal{P}(\emptyset) = \{P_\alpha(\emptyset) : \alpha \in \Lambda\}$.

For a finite sequence δ of $\mathbb{N}$, if $\mathcal{P}(\delta)$ has been defined and $n \in \mathbb{N}$, we shall define $\mathcal{P}(\delta n)$ as follows:

Denote $\mathcal{P}(\delta)$ by $\{P_\alpha(\delta) : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, put

$P^*_\alpha(\delta n) = F(n, X \setminus \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\delta)})$,

$P_\alpha(\delta n) = P^*_\alpha(\delta n) \setminus \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\delta n)}$, and

$\mathcal{P}(\delta n) = \{P_\alpha(\delta n) : \alpha \in \Lambda\}$.

Let $U_\alpha = \cup\{P_\alpha(\delta) : \delta$ is a finite sequence of $\mathbb{N}\}$, $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$. We shall show that $\mathcal{U}$ is the desired family.

First, $F_\alpha = P_\alpha(\emptyset) \subset U$ for each $\alpha \in \Lambda$.

Since X is a k-space with a point-G_δ property, X is a sequential space [11]. To show that each U_α is open, it suffices to show that U_α is a sequential neighborhood of U_α. Let S be a sequence converging to $x \in U_\alpha$. There is a finite sequence δ of $\mathbb{N}$ such that $x \in P_\alpha(\delta)$. Put $M_\alpha = \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\delta)}$. Then $x \notin M_\alpha$, and thus S is eventually in $F(m, X \setminus M_\alpha) = P^*_\alpha(\delta m)$ for some $m \in \mathbb{N}$. If $x \in \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\delta m)}$, then $x \in \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} F(m, X \setminus M_\beta)} \subset F(m, \bigcup_{\beta\in\Lambda\setminus\{\alpha\}}(X \setminus M_\beta)) \subset \bigcup_{\beta\in\Lambda\setminus\{\alpha\}}(X \setminus M_\beta)$; thus, $x \in X \setminus M_\beta \subset X \setminus P^*_\alpha(\delta) \subset X \setminus P_\alpha(\delta)$ for some $\beta \in \Lambda\setminus\{\alpha\}$, a contradiction. Hence, $x \in X \setminus \overline{\bigcup_{\beta\in\Lambda\setminus\{\alpha\}} P^*_\beta(\delta m)}$, and S is eventually in $P_\alpha(\delta m) \subset U_\alpha$, so U_α is a sequential neighborhood of U_α.

If $\mathcal{U}$ is not point-countable, then $|\{\alpha \in \Lambda : x \in U_\alpha\}| > \omega$ for some $x \in X$; thus, there are a finite sequence δ of $\mathbb{N}$ and an uncountable subset Λ' of Λ such that $x \in P_\alpha(\delta)$ for each $\alpha \in \Lambda'$, a contradiction because $\{P_\alpha(\delta) : \alpha \in \Lambda\}$ is disjoint.

(2) X is a meta-Lindelöf space.

Let $\mathcal{W}$ be an open cover of the space X. By the subparacompactness of X, $\mathcal{W}$ has a refinement $\bigcup_{i\in\mathbb{N}} \mathcal{F}_i$, where each $\mathcal{F}_i = \{F_{i\alpha} : \alpha \in \Lambda_i\}$ is a discrete family of closed subsets of X. For each $i \in \mathbb{N}$, $\mathcal{F}_i$ may be expanded to a point-countable family $\mathcal{U}_i = \{U_{i\alpha} : \alpha \in \Lambda_i\}$ of open subsets of X from (1). Take $W_{i\alpha} \in \mathcal{W}$ such that $F_{i\alpha} \subset W_{i\alpha}$ for each $\alpha \in \Lambda_i$. Then $\bigcup_{i\in\mathbb{N},\alpha\in\Lambda_i} W_{i\alpha} \cap U_{i\alpha}$ is a point-countable open refinement of $\mathcal{W}$; thus, X is a meta-Lindelöf space. □

To find a paracompactness of k-semistratifiable spaces, we state a fine k-semistratification.

Lemma 1. *Let X be a k-semistratifiable space. Then for each subset W of X there is a sequence $\{H(n, W)\}_{n\in\mathbb{N}}$ of closed subsets of X such that*

(1) $H(n, W) \subset H(n + 1, W) \subset W$;

(2) if $V \subset W$, then $H(n, V) \subset H(n, W)$;

(3) if W is a sequential neighborhood of x, then every sequence converging to x is eventually in $H(m, W)$ for some $m \in \mathbb{N}$;

(4) if $\{G_\alpha : \alpha \in \Lambda\}$ is a disjoint family of subsets of X and $n \in \mathbb{N}$, then $\{H(n, G_\alpha) : \alpha \in \Lambda\}$ is a discrete family in X.

Proof: Let $U \to \{F(n, U)\}_{n\in\mathbb{N}}$ be a k-semistratification for X. We can assume that each $F(n, U) \subset F(n + 1, U)$. For each $n \in \mathbb{N}$

and $x \in X$, define that $g(n,x) = X \setminus F(n, X \setminus \{x\})$, then $g(n,x)$ is open in X and $x \in g(n+1,x) \subset g(n,x)$. For each $n \in \mathbb{N}$ and $W \subset X$, put $H(n,W) = X \setminus \bigcup_{x \in X \setminus W} g(n,x)$, then $H(n,W)$ is closed in X and satisfies conditions (1) and (2).

Let W be a sequential neighborhood of x in X and a sequence $\{x_n\}$ converges to x. If (3) does not hold, then for each $i \in \mathbb{N}$, there is $x_{n_i} \in X \setminus H(i,W)$; thus, there is $y_i \in X \setminus W$ such that $x_{n_i} \in g(i,y_i)$. Let U be an open neighborhood of x. There are $k, m \in \mathbb{N}$ such that $\{x_{n_i} : i \geq k\} \subset F(m,U)$; thus, $y_i \in U$ for each $i \geq k$, and hence the sequence $\{y_i\}$ converges to x, a contradiction because W is a sequential neighborhood of x.

Let $\{G_\alpha : \alpha \in \Lambda\}$ be a disjoint family of subsets of X and $n \in \mathbb{N}$. For each $x \in X$, take $V = X \setminus H(n, \cup\{G_\alpha : \alpha \in \Lambda \text{ and } x \notin G_\alpha\})$, then V is an open neighborhood of x in X and $V \cap H(n, G_\alpha) = \emptyset$ if $x \notin G_\alpha$. Hence, $\{H(n,G_\alpha) : \alpha \in \Lambda\}$ is a discrete family of subsets of X. □

Theorem 2. *Every k-semistratifiable, normal k-space is a hereditarily paracompact space.*

Proof: Let X be a k-semistratifiable, normal k-space, and $W \to \{H(n,W)\}_{n \in \mathbb{N}}$ a correspondence of X satisfying all conditions in Lemma 1. By Definition 2, X is a perfect space (i. e., a space in which each closed subset is a G_δ-set). It is easy to check that a perfect paracompact space is a hereditarily paracompact space [1]. To complete the proof, it suffices to show that X is a paracompact space.

Let $\mathcal{F}$ be a discrete family of closed subsets of X.

(1) $\mathcal{F}$ may be expanded to a disjoint family of sequential neighborhoods.

Put $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, let $L_\alpha = \bigcup_{\beta \in \Lambda \setminus \{\alpha\}} F_\beta$ and $G_\alpha = \bigcup_{n \in \mathbb{N}} (H(n, X \setminus L_\alpha) \setminus H(n, X \setminus F_\alpha))$. Then $\{G_\alpha : \alpha \in \Lambda\}$ is a disjoint family of subsets of X. We shall prove that each G_α is a sequential neighborhood of F_α.

Let S be a sequence converging to $x \in F_\alpha$. Since L_α is closed and $F_\alpha \cap L_\alpha = \emptyset$, S is eventually in $H(m, X \setminus L_\alpha)$ for some $m \in \mathbb{N}$, and $x \notin H(m, X \setminus F_\alpha)$, so we may assume that S is eventually in $H(m, X \setminus L_\alpha) \setminus H(m, X \setminus F_\alpha) \subset G_\alpha$; hence, G_α is a sequential neighborhood of F_α.

(2) $\mathcal{F}$ may be expanded to a discrete family of closed sequential neighborhoods.

For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$, $H(n, X \setminus F_\alpha) \cap F_\alpha = \emptyset$. By normality, there is an open subset $V_\alpha(n)$ such that $F_\alpha \subset V_\alpha(n) \subset \overline{V_\alpha(n)} \subset X \setminus H(n, X \setminus F_\alpha)$. Put $F_\alpha(n) = H(n, G_\alpha) \cap \overline{V_\alpha(n)}$ and $W_\alpha = \bigcup_{n\in\mathbb{N}} F_\alpha(n)$. Then W_α is a sequential neighborhood of F_α. In fact, if S is a sequence converging to $x \in F_\alpha$, S is eventually in $H(m, G_\alpha)$ for some $m \in \mathbb{N}$ by (1) and Lemma 1, and S is eventually in $V_\alpha(m)$; thus, S is eventually in $F_\alpha(m) \subset W$.

Let $\mathcal{W} = \{W_\alpha : \alpha \in \Lambda\}$. Then $\mathcal{W}$ is a disjoint family because of each $W_\alpha \subset G_\alpha$. To show that $\mathcal{W}$ is a discrete family of closed subsets, it suffices to show that $\mathcal{W}$ is a closure-preserving family of closed subsets, i. e., $\bigcup_{\alpha\in\Lambda'} W_\alpha$ is closed in X for each $\Lambda' \subset \Lambda$. Since X is a k-space with a point-G_δ property, X is a sequential space. Let S be a sequence converging to $x \notin \bigcup_{\alpha\in\Lambda'} W_\alpha$. Then $x \notin \bigcup_{\alpha\in\Lambda'} F_\alpha$, S is eventually in $H(m, X \setminus \bigcup_{\alpha\in\Lambda'} F_\alpha)$ for some $m \in \mathbb{N}$, and $H(m, X \setminus F_\alpha) \cap F_\alpha(n) = \emptyset$ for each $\alpha \in \Lambda'$ and $n \geq m$. By Lemma 1, $\{H(n, G_\alpha) : \alpha \in \Lambda\}$ is a discrete family in X for each $n \in \mathbb{N}$, so $\{F_\alpha(n) : \alpha \in \Lambda\}$ is a discrete family of closed subsets of X. Put $E(m, \Lambda') = \bigcup_{\alpha\in\Lambda', n<m} F_\alpha(n)$. Then $E(m, \Lambda')$ is closed and $x \notin E(m, \Lambda')$; thus, S is eventually in $X \setminus E(m, \Lambda')$. Hence, S is eventually in $X \setminus \bigcup_{\alpha\in\Lambda'} W_\alpha$, and $\bigcup_{\alpha\in\Lambda'} W_\alpha$ is closed in X.

(3) X is a collectionwise normal space.

Let $\mathcal{H}_1$ be a discrete family of closed subsets of X. By (2), there is a sequence $\{\mathcal{H}_n\}$ of discrete families of closed subsets of X such that each $\mathcal{H}_{n+1}$ is an expansion of sequential neighborhoods of $\mathcal{H}_n$. Index $\mathcal{H}_n$ by $\{H_\alpha(n) : \alpha \in \Lambda\}$ for each $n \in \mathbb{N}$. Put $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$, where each $H_\alpha = \bigcup_{n\in\mathbb{N}} H_\alpha(n)$. Suppose that S is a sequence converging to $x \in H_\alpha$. Then $x \in H_\alpha(j)$ for some $j \in \mathbb{N}$. Since $H_\alpha(j+1)$ is a sequential neighborhood of $H_\alpha(j)$, S is eventually in $H_\alpha(j+1) \subset H_\alpha$; hence, H_α is open in X. Therefore, $\mathcal{H}$ is a disjoint open expansion of $\mathcal{H}_1$, and X is a collectionwise normal space.

Since X is a subparacompact space, X is a paracompact space from (3). □

Zhi Min Gao [7] proved that a normal space with a σ-closure-preserving weak base is a paracompact space. It can be shown that a regular space with a σ-closure-preserving weak base is a meta-Lindelöf space by a similar technique in [7] and Theorem 1. The author doesn't know whether a regular k-space with a σ-locally finite k-network (i.e., a k- and $\aleph$-space) has a σ-closure-preserving weak base.

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