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On the Countable Tightness of Product Spaces

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Abstract In this paper, we discuss the countable tightness of products of spaces which are quotient s-images of locally separable metric spaces, or k-spaces with a star-countable k-network. The main result is that the following conditions are equivalent: (1) $\mathbf{b} = \omega_1$; (2) $t(S_\omega \times S_{\omega_1}) > \omega$; (3) For any pair (X, Y), which are k-spaces with a point-countable k-network consisting of cosmic subspaces, $t(X \times Y) \le \omega$ if and only if one of X, Y is first countable or both X, Y are locally cosmic spaces. Many results on the k-space property of products of spaces with certain k-networks could be deduced from the above theorem.

Keywords Countable tightness, k-spaces, Cosmic spaces, Product spaces, k-networks, Point-countable collections, Star-countable collections

MR(2000) Subject Classification 54D50, 54B10, 54A25, 54E20

1 Introduction

All spaces are regular and T_1, all maps are continuous and onto. The countable tightness of product spaces was investigated in [1], [2] and [3], and others. In this paper we consider the countable tightness of products of k-spaces with certain k-networks. By using results on countable tightness of product spaces, we give a necessary and sufficient condition for the products of k-spaces having a point-countable k-network with some properties to be a k-space.

It is well known that the concept of "base" is one of the most important concepts for topological spaces. As a generalization of "base", the notion of "k-network" was introduced by O'Meara [4]. He used this concept to discuss paracompactness in function spaces with the compact-open topology. In this paper we shall show some applications of "k-network" about countable tightness and k-space property of product spaces. Let us recall some definitions.

Definition 1.1 *Let $\mathscr{P}$ be a cover of a space X.*

$\mathscr{P}$ is called a network for X if for any point $x \in X$, and for any open set U with $x \in U$, $x \in P \subset U$ for some $P \in \mathscr{P}$. $\mathscr{P}$ is called a k-network for X if for any compact set K of X, and for any open set U with $K \subset U$, $K \subset \bigcup \mathscr{F} \subset U$ for some finite $\mathscr{F} \subset \mathscr{P}$.

A cosmic space is a space with a countable network. An $\aleph_0$-space is a space with a countable k-network. An $\aleph$-space is a space with a σ-locally finite k-network.

Definition 1.2 *Let $\mathscr{F}$ be a cover of a space X. X is determined by $\mathscr{F}$ if $A \subset X$ is closed whenever $A \cap F$ is closed in F for each $F \in \mathscr{F}$.*

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A k-space (resp. sequential space) is a space determined by the cover of all compact subsets (resp. compact metric subsets). A k_ω-space is a space determined by a countable cover of compact subsets.

It is known that the product of two k-spaces need not be a k-space. Michael [5] posed the following natural problem: "*Let X and Y be k-spaces. Find the sufficient and necessary conditions for $X \times Y$ to be a k-space.*" A partial answer to the above problem is given by Tanaka. Let X and Y be topological spaces. We say that the pair (X, Y) satisfies the *Tanaka condition* if one of the following conditions is satisfied:

(a) Both X and Y are first countable;

(b) One of X, Y is locally compact, and the other is a k-space;

(c) Both X and Y are local k_ω-spaces.

It is easy to see that $X \times Y$ is a k-space if (X, Y) satisfies the Tanaka condition. Tanaka [6; Theorem 3.1] got a sufficient and necessary condition for product spaces on k-and $\aleph$-spaces.

Theorem 1.3 *Let X and Y be k-and $\aleph$-spaces. Then $X \times Y$ is a k-space if and only if the pair (X, Y) satisfies the Tanaka condition.*

To state further results about Michael's problem, we recall a set-theoretic axiom. Let ω^ω be the set of all functions from ω to ω. For $f, g \in \omega^\omega$, define $g \leq^* f$ if $\{n \in \omega : f(n) < g(n)\}$ is finite. Let $\mathbf{b} = \min\{\gamma$: there is a $\leq^*$-unbounded family $A \subset \omega^\omega$ with $|A| = \gamma\}$, where A is *"$\leq^*$-unbounded"* if and only if no $f \in \omega^\omega$ is $^*\geq$ every $g \in A$. By BF(ω_2), we mean "$\mathbf{b} \geq \omega_2$". It is known that: (1) $\mathbf{b} \geq \omega_1$; (2) (CH)$\Rightarrow \neg$BF$(\omega_2) \Leftrightarrow$ "$\mathbf{b} = \omega_1$"; (3) (MA)$\Rightarrow$ "$\mathbf{b} = 2^\omega$" [7].

A closed image of a metric space is called a *Lašnev space*. Tanaka [8, Theorem 1.1] proved that Theorem 1.3 remains valid if X, Y are Lašnev spaces under (CH). Gruenhage [9, Theorem 1] improved Tanaka's result. Let α be an infinite cardinal number and S_α the quotient space obtained from the disjoint union of α convergent sequences by identifying all limit points. The S_α is a Lašnev space, which is called a *fan space*, in particular, S_ω is called a *sequential fan*. Let $\mathscr{P} = \{P_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. $\mathscr{P}$ is *compact-countable* if each compact subset of X meets at most countably many P_α. Every Lašnev space has a compact-countable k-network. Liu and Tanaka investigated the k-space property of the products of spaces with certain k-networks ([10, Theorem 2.6], [11, Theorem 3.4]) and the generalized Gruenhage's result as follows.

Theorem 1.4 *The following conditions are equivalent:*

(1) $\mathbf{b} = \omega_1$;

(2) $S_\omega \times S_{\omega_1}$ *is not a k-space*;

(3) *For any pair (X, Y), where X, Y have a compact-countable k-network, $X \times Y$ is a k-space if and only if (X, Y) satisfies the Tanaka condition.*

Definition 1.5 *A space X has countable tightness (simply, $t(X) \leq \omega$) if, whenever $x \in \mathrm{cl}A$, $x \in \mathrm{cl}B$ for some countable subset B of A.*

As is well known, $t(X) \leq \omega$ if and only if X is a space determined by the cover of all countable subsets. Every hereditary separable space or sequential space has countable tightness. Every subspace of spaces with countable tightness has countable tightness. The countable tightness is preserved by quotient mappings [12]. Let $\mathscr{P} = \{P_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. $\mathscr{P}$ is *point-countable* if each point of X is contained in at most countably many P_α. Gruenhage, Michael and Tanaka [13, Corollary 3.4] proved that any k-space with a point-countable k-network is a sequential space, and hence has countable tightness. Therefore we have the following problem:

Problem 1.6 *Let X, Y be spaces with certain k-networks. What are the necessary and sufficient conditions for the product $X \times Y$ to have countable tightness?*

The main result in this paper is the following theorem, which gives a partial answer to the above problem and deduces many results on the k-space property of products of spaces with certain k-networks:

Theorem 1.7 *The following conditions are equivalent*:

(1) $\mathbf{b} = \omega_1$;

(2) $t(S_\omega \times S_{\omega_1}) > \omega$;

(3) *For any pair* (X, Y)*, which are* k*-spaces with a point-countable* k*-network consisting of cosmic subspaces,* $t(X \times Y) \le \omega$ *if and only if*: (*a*) X *or* Y *is first countable*; *or,* (*b*) *both* X *and* Y *are locally cosmic spaces.*

At the end of this section we recall the *Arens space* S_2. Let $S_2 = (\mathbb{N}\times\mathbb{N}) \cup \mathbb{N} \cup \{0\}$ endowed with the following topology: Each point of $\mathbb{N}\times\mathbb{N}$ is an isolated point; a basis of neighborhood of $n \in \mathbb{N}$ consists of all sets of the form $\{n\} \cup \{(m, n) : m \ge k\}$ for each $k \in \mathbb{N}$; and U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in \mathbb{N}$. Obviously, S_ω is a perfect image of S_2. Thus for a space X, if $t(S_2 \times X) \le \omega$, then $t(S_\omega \times X) \le \omega$; and $S_2 \times X$ is a k-space if and only if $S_\omega \times X$ is a k-space.

2 Countable Tightness of Products of Spaces with Certain k-networks

Let κ be a cardinal number. Let X be a space with $|X| \ge \kappa$, and $X_1 \subset X$ with $|X_1| = \kappa$. We write $X_1 = \{x_\alpha : \alpha < \kappa\}$. Let $L_\alpha = \{y_\alpha(n) : n \in \mathbb{N}\}$, $L_\alpha \cap X = \emptyset$, $L_\alpha \to y^\alpha$ for each $\alpha < \kappa$, $y^\alpha \ne y^\beta$ and $L_\alpha \cap L_\beta = \emptyset$ if $\alpha \ne \beta$. $X(\kappa) = X \cup \{L_\alpha : \alpha < \kappa\}$ is the quotient space by identifying x_α and y^α with a point for each $\alpha < \kappa$. We can see that each point in $\bigcup\{L_\alpha : \alpha < \kappa\}$ is an isolated point, and for $x \in X$, the basic neighborhood of x in $X(\kappa)$ consists of all sets of the form $U \cup \{y_\alpha(n) : n \ge m(\alpha), \alpha \in \{\gamma : x_\gamma \in U\}\}$, where $m(\alpha) \in \mathbb{N}$ and U is a neighborhood of x in X. Put:

$\mathscr{T}(\kappa) = \{X(\kappa) : X$ has countable tightness, and $|X| \ge \kappa\}$;

$\mathscr{L}(\kappa) = \{X(\kappa) \in \mathscr{T}(\kappa) : X$ is a Lindelöf space$\}$.

The following lemmas are the modifications of Lemmas and Theorems in [9] and [2]:

Lemma 2.1 (1) $t(S_\omega \times Y) > \omega$ *for each* $Y \in \mathscr{L}(\mathbf{b})$.

(2) *For* $\omega \le \kappa < \mathbf{b}$, $t(S_\omega \times X(\kappa)) \le \omega$ *if and only if* $t(S_\omega \times X) \le \omega$.

Proof (1) We write $S_\omega = \{x_0\} \cup \{x_n(i) : i, n \in \mathbb{N}\}$, where $x_n(i) \to x_0$ for each $n \in \mathbb{N}$; $Y = X \cup \{L_\alpha : \alpha < \mathbf{b}\}$, where $L_\alpha = \{y_\alpha(n) : n \in \mathbb{N}\}$. For each $\alpha < \mathbf{b}$, let $H_\alpha = \{(x_n(f_\alpha(n)), y_\alpha(i)) : i \le n \in \mathbb{N}\}$, where $f_\alpha \in B$, $|B|$=$\mathbf{b}$, and B is a $\le^*$-unbounded subset of ω^ω. Let $H = \bigcup\{H_\alpha : \alpha < \mathbf{b}\}$. Put $Z = \{x_0\} \times X$. Then $Z \cap \mathrm{cl}H \ne \emptyset$.

Otherwise, for each $(x_0, y) \in Z$, choose a neighborhood $U_{f_y} \times V_y$ of (x_0, y) such that $(U_{f_y} \times V_y) \cap H = \emptyset$, where $f_y \in \omega^\omega$, $U_{f_y} = \{x_0\} \cup \{x_n(j) : j > f_y(n), n \in \mathbb{N}\}$, and V_y is a basic neighborhood of y in Y. $\{V_y : y \in X\}$ is an open cover of a Lindelöf space X, so there exists a countable subfamily $\{V_{y_i} : i \in \mathbb{N}\}$ which covers X. Since $\mathbf{b}$ is a regular cardinal, there is $k \in \mathbb{N}$ such that $|\{\alpha < \mathbf{b} : L_\alpha \cap V_{y_k} \ne \emptyset\}| = \mathbf{b}$ and $\{f_\gamma : \gamma \in A\}$ is $\le^*$-unbounded, where $A = \{\alpha < \mathbf{b} : L_\alpha \cap V_{y_k} \ne \emptyset\}$. We can pick $\beta \in A$ such that $|\{n \in \mathbb{N} : f_\beta(n) > f_{y_k}(n)\}| = \omega$. Then $(U_{f_{y_k}} \times V_{y_k}) \cap H_\beta \ne \emptyset$. In fact, there exists $n_0 \in \{n \in \mathbb{N} : f_\beta(n) > f_{y_k}(n)\}$ such that $y_\beta(n_0) \in V_{y_k}$. Obviously, $x_{n_0}(f_\beta(n_0)) \in U_{f_{y_k}}$, so $(x_{n_0}(f_\beta(n_0)), y_\beta(n_0)) \in H_\beta$. This is a contradiction.

Let $z \in Z \cap \mathrm{cl}H$. If $t(S_\omega \times Y) \le \omega$, there is a countable subset C of H such that $z \in \mathrm{cl}C$. There exists a sequence $\{\alpha_i\}$ of distinct ordinal numbers such that $C \subset \bigcup\{H_{\alpha_i} : i \in \mathbb{N}\}$. Pick $f \in \omega^\omega, f(n) > f_{\alpha_i}(k)$ for $i, k \le n \in \mathbb{N}$. Let $V = Y \setminus \bigcup\{A_n : n \in \mathbb{N}\}$, where $A_n = \{y_{\alpha_n}(k) : k \le n\}$. Then V is open. Let $W = U_f \times V$; then W is open and $z \in W$, thus $W \cap C \ne \emptyset$, so $W \cap H_{\alpha_m} \ne \emptyset$ for some $m \in \mathbb{N}$, hence $(x_n(f_{\alpha_m}(n)), y_{\alpha_m}(i)) \in U_f \times V$ for some $i \le n$. Thus $x_n(f_{\alpha_m}(n)) \in U_f, f_{\alpha_m}(n) > f(n)$, so $m > n$. On the other hand, $y_{\alpha_m}(i) \in V$, which means $i > m$, hence $n > m$, a contradiction. So $t(S_\omega \times Y) > \omega$.

As for (2), we may use a method similar to that in [2] to prove it. Here we omit the proof.

For an infinite cardinal number α, let $A_\alpha = (\bigcup\{C_\beta : \beta < \alpha\}) \cup \{c\}$ be a space which satisfies the following: All C_β's are countable sets which are pairwise disjoint, $c \in \mathrm{cl}C_\beta \setminus C_\beta$ for all β, and for each finite $F_\beta \subset C_\beta$, and a subset $E \subset \alpha$, $\bigcup\{F_\beta : \beta \in E\}$ is closed in A_α. If each C_β is a sequence converging to c, and all points except for c are isolated, then A_α is homeomorphic to the fan space S_α.

Lemma 2.2 $t(S_\omega \times A_{\mathbf{b}}) > \omega$.

Proof Let B be a $\leq^*$-unbounded subset of ω^ω with $|B| = \mathbf{b}$. We write $S_\omega = \{x\} \cup \{x_n(i) : i, n \in \mathbb{N}\}$, where the sequence $x_n(i) \to x$ for each $n \in \mathbb{N}$. For each $\beta < \mathbf{b}$, let $H_\beta = \{(x_n(f_\beta(n)), c_\beta(i)) : i \leq n \in \mathbb{N}\}$, where $C_\beta = \{c_\beta(i) : i \in \mathbb{N}\}$, $f_\beta \in B$. Put $H = \bigcup\{H_\beta : \beta < \mathbf{b}\}$. In view of the proof of (1) of Lemma 2.1, we can see that $(x, c) \in \mathrm{cl}H$ but there is no countable subset of H whose closure contains (x, c). So $\mathrm{t}(S_\omega \times A_{\mathbf{b}}) > \omega$.

Lemma 2.3 *$t(Y^2) > \omega$ for each $Y \in \mathscr{L}(\omega_1)$.*

Proof Let $Y = X \cup \{L_\alpha : \alpha < \omega_1\}$, where $L_\alpha = \{y_\alpha(n) : n \in \mathbb{N}\} \to y^\alpha$. For each $\alpha < \omega_1$, let $f_\alpha : \omega_1 \to \omega$ be a function such that f_α restricted to α is a one-to-one map onto ω when $\alpha \geq \omega$. Put $H_\alpha = \{(y_\beta(f_\alpha(\beta)), y_\alpha(f_\alpha(\beta))) : \beta < \omega_1\}$, and let $H = \bigcup\{H_\alpha : \alpha < \omega_1\}$. Then $X^2 \cap \mathrm{cl}H \neq \emptyset$.

Suppose this is not the case. Then for each $x \in X$, there exists a basic neighborhood $U(x)$ of x in Y such that $U(x)^2 \cap H = \emptyset$. $\{U(x) : x \in X\}$ is an open cover of a Lindelöf space X, and there is a countable subcover $\{U(x_i) : i \in \mathbb{N}\}$. Here there exists $x' \in X$ such that $|\{\alpha < \omega_1 : U(x') \cap L_\alpha \neq \emptyset\}| = \omega_1$. Thus there is $n_0 \in \mathbb{N}$, and $A \subset \{\alpha < \omega_1 : U(x') \cap L_\alpha \neq \emptyset\}$ with $|A| = \omega_1$ such that $y_\alpha(m) \in U(x') \cap L_\alpha$ for any $m \geq n_0$ and $\alpha \in A$. Pick $\gamma \in A$ such that γ has infinitely many predecessors in A. Since $f_\gamma : \gamma \to \omega$ is one-to-one, there must be $\delta \in A, \delta < \gamma$ such that $f_\gamma(\delta) > n_0$. It is clear that $(y_\delta(f_\gamma(\delta)), y_\gamma(f_\gamma(\delta))) \in U(x')^2 \cap H_\gamma$, a contradiction.

Let B be a countable subset of H. There exists a sequence $\{\alpha_i\}$ such that $B \subset \bigcup\{H_{\alpha_i} : i \in \mathbb{N}\}$, and so $B \subset \{(y_{\beta_i}(f_{\alpha_j}(\beta_i)), y_{\alpha_j}(f_{\alpha_j}(\beta_i))) : i, j \in \mathbb{N}\}$. For $n \in \mathbb{N}$, put:

$L'_{\beta_n} = L_{\beta_n} \setminus \{y_{\beta_n}(j) : j \leq k_n\}$, where $k_n = \max\{f_{\alpha_j}(\beta_n) : j \leq n\}$;

$L''_{\alpha_n} = L_{\alpha_n} \setminus \{y_{\alpha_n}(i) : i \leq l_n\}$, where $l_n = \max\{f_{\alpha_n}(\beta_i) : i \leq n\}$.

And let:

$U = (Y \setminus \cup\{L_{\beta_n} : n \in \mathbb{N}\}) \bigcup (\cup\{L'_{\beta_n} : n \in \mathbb{N}\})$;

$V = (Y \setminus \cup\{L_{\alpha_n} : n \in \mathbb{N}\}) \bigcup (\cup\{L''_{\alpha_n} : n \in \mathbb{N}\})$.

Then $X \subset U \cap V$, and $X^2 \subset U \times V$. But $B \cap (U \times V) = \emptyset$. In fact, $y_{\beta_i}(f_{\alpha_j}(\beta_i)) \in U$ means $j > i$, and $y_{\alpha_j}(f_{\alpha_j}(\beta_i)) \in V$ means $i > j$. Thus $X^2 \cap \mathrm{cl}B = \emptyset$. Therefore, $t(Y^2) > \omega$.

Let $\mathscr{P} = \{P_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. $\mathscr{P}$ is *star-countable* if each P_α meets at most countably many elements of $\mathscr{P}$. Let $\mathbb{I} = [0, 1]$ be the closed unit interval. For an infinite cardinal number κ, $\mathbb{I}(\kappa)$ is a quotient and compact image of a locally separable metric space and has a star-countable k-network, see [10] for example. Also, it is known that any separable subspace in $\mathbb{I}(\kappa)$ is an $\aleph_0$-space [14; Corollary 2.5]. The fan space S_κ is homeomorphic to the quotient space obtained from $\mathbb{I}(\kappa)$ by identifying $\mathbb{I}$ with a point. Thus S_κ is a perfect image of the space $\mathbb{I}(\kappa)$.

Remark 2.4 (1) $t(S_\omega \times \mathbb{I}(\mathbf{b})) > \omega$ by Lemma 2.1.

(2) If $\mathbf{b} > \omega_1$, then $t(S_\omega \times \mathbb{I}(\omega_1)) \leq \omega$ by Lemma 2.1.

(3) Let $\omega \leq \kappa < \mathbf{b}$. Then $t(S_\omega \times S_\kappa) \leq \omega$.

(4) $\mathrm{t}(S_\omega \times S_{\mathbf{b}}) > \omega$ by Lemma 2.2.

(5) $t((S_{\omega_1})^2) > \omega$ by Lemma 2.3.

Lemma 2.5 *Let X be a k-space with a point-countable k-network. Then X has a point-countable base if X contains no closed copy of either S_ω or S_2* [15; *Corollary* 3.9].

We call W a *sequential neighborhood* of A if any sequence L which converges to a point of A is eventually in W.

Theorem 2.6 *The following conditions are equivalent*:

(1) $\mathbf{b} = \omega_1$;

(2) $t(S_\omega \times S_{\omega_1}) > \omega$;

(3) *For any pair* (X, Y)*, which are* k*-spaces with a point-countable* k*-network consisting of cosmic subspaces,* $t(X \times Y) \leq \omega$ *if and only if*: (a) X *or* Y *is first countable*; *or,* (b) *both* X *and* Y *are locally cosmic spaces.*

Proof (1) $\Rightarrow$ (3). "Only if": $t(S_\omega \times Z) > \omega$ for $Z \in \mathscr{L}(\omega_1)$ by Lemma 2.1, and $t(S_\omega \times S_{\omega_1}) > \omega$ by Remark 2.4.

Suppose neither X nor Y is first-countable. Then X and Y contain a closed copy of either S_ω or S_2 by Lemma 2.5. Without loss of generality, we assume that X and Y contain a closed copy of S_ω. Let $\mathscr{B}$ be a point-countable k-network of Y, and let each element of $\mathscr{B}$ be a cosmic subspace.

Claim *For* $y \in Y$*, there is a countable subfamily* $\mathscr{B}_1 \subset \mathscr{B}$ *such that* $\cup\mathscr{B}_1$ *is a sequential neighborhood of* y*.*

Suppose this is not the case. For a sequence $y^1(n) \to y$, let $\mathscr{P}_1 = \{P \in \mathscr{B} : P \cap (\{y\} \cup \{y^1(n) : n \in \mathbb{N}\}) \neq \emptyset\}$. Then $|\mathscr{P}_1| \leq \omega$. Since $\cup\mathscr{P}_1$ is not a sequential neighborhood of y, there exists a sequence $y^2(n) \to y$ such that $\{y^2(n) : n \in \mathbb{N}\} \cap (\cup\mathscr{P}_1) = \emptyset$. Let $\mathscr{P}_2 = \{P \in \mathscr{B} : P \cap \{y^2(n) : n \in \mathbb{N}\} \neq \emptyset\}$. Then $|\mathscr{P}_2| \leq \omega$. Since $\cup(\mathscr{P}_1 \cup \mathscr{P}_2)$ is not a sequential neighborhood of y, there is a sequence $y^3(n) \to y$ such that $\{y^3(n) : n \in \mathbb{N}\} \cap (\cup(\mathscr{P}_1 \cup \mathscr{P}_2)) = \emptyset$. Let $\mathscr{P}_3 = \{P \in \mathscr{B} : P \cap \{y^3(n) : n \in \mathbb{N}\} \neq \emptyset\}$. Clearly $|\mathscr{P}_3| \leq \omega$. Inductively, we can get a sequence $\{\{y^\alpha(n) : n \in N\} : \alpha < \omega_1\}$ such that each element of $\mathscr{B}$ meets at most one $\{y^\alpha(n) : n \in \mathbb{N}\}$. Since Y is a sequential space having a k-network $\mathscr{B}$, it is not difficult to check that $\{y\} \cup \{y^\alpha(n) : \alpha < \omega_1,$ and $n \in \mathbb{N}\}$ is a copy of S_{ω_1}. But $t(S_\omega \times Y) \leq \omega$, so $t(S_\omega \times S_{\omega_1}) \leq \omega$, a contradiction.

So there is a countable subfamily $\mathscr{B}_1$ of $\mathscr{B}$ such that $\cup\mathscr{B}_1$ is a sequential neighborhood of y. Also, $\cup\mathscr{B}_1$ is a cosmic space, hence it is a hereditary Lindelöf space.

Assuming that $\mathscr{B}_n$ has been selected out, where $\cup\mathscr{B}_n$ is a sequential neighborhood of $\cup\mathscr{B}_{n-1}$ and is a Lindelöf space, we can choose a countable subfamily $\mathscr{B}_{n+1} \subset \mathscr{B}$ such that $\cup\mathscr{B}_{n+1}$ is a sequential neighborhood of $\cup\mathscr{B}_n$.

Suppose this is not the case. For a sequence $z^1(n) \to z_1 \in \cup\mathscr{B}_n$, let $\mathscr{Q}_1 = \{P \in \mathscr{B} : P \cap (\{z^1(n) : n \in \mathbb{N}\} \cup \{z_1\}) \neq \emptyset\}$. Then $|\mathscr{Q}_1| \leq \omega$. Since $\cup\mathscr{Q}_1$ is not a sequential neighborhood of $\cup\mathscr{B}_n$, there is a sequence $z^2(n) \to z_2 \in \cup\mathscr{B}_n$ such that $\{z^2(n) : n \in \mathbb{N}\} \cap (\cup\mathscr{Q}_1) = \emptyset$. Let $\mathscr{Q}_2 = \{P \in \mathscr{B} : P \cap (\{z^2(n) : n \in \mathbb{N}\} \cup \{z_2\}) \neq \emptyset\}$, then $|\mathscr{Q}_2| \leq \omega$. Since $\mathscr{Q}_1 \cup \mathscr{Q}_2$ is a countable subfamily of $\mathscr{B}$, then there is a sequence $z^3(n) \to z_3 \in \cup\mathscr{B}_n$ such that $\{z^3(n) : n \in \mathbb{N}\} \cap (\cup(\mathscr{Q}_1 \cup \mathscr{Q}_2)) = \emptyset$. Let $\mathscr{Q}_3 = \{P \in \mathscr{B} : P \cap (\{z^3(n) : n \in \mathbb{N}\} \cup \{z_3\}) \neq \emptyset\}$. Then $|\mathscr{Q}_3| \leq \omega$. In this way, we can obtain a sequence $\{\{z^\alpha(n) : n \in \mathbb{N}\} : \alpha < \omega_1\}$ such that $z^\alpha(n) \to z_\alpha$, and each element of $\mathscr{B}_n$ meets at most one $\{z^\alpha(n) : n \in \mathbb{N}\}$.

Case 1 If $|\{z_\alpha : \alpha < \omega_1\}| \leq \omega$, then Y must contain a copy of S_{ω_1}, a contradiction.

Case 2 If $|\{z_\alpha : \alpha < \omega_1\}| > \omega$, then Y contains a subspace $Y_1 \in \mathscr{L}(\omega_1)$, a contradiction.

By recursion, we can get a sequence $\{\mathscr{B}_n\}$ such that each $\cup\mathscr{B}_{n+1}$ is a sequential neighborhood of $\cup\mathscr{B}_n$ and is a cosmic subspace of Y.

Let $V = \cup\{\cup\mathscr{B}_n : n \in \mathbb{N}\}$. Then V is a sequential open set containing y and V is a cosmic subspace. Since Y is a sequential space, V is a neighborhood of y. This implies that Y is locally cosmic.

"If": If one of X, Y is first countable, then $t(X \times Y) \leq \omega$ ([1]). If both X and Y are locally cosmic, it is clear that $t(X \times Y) \leq \omega$.

(3) $\Rightarrow$ (2) S_ω, S_{ω_1} are k-spaces with a point-countable k-network consisting of cosmic subspaces [16, Lemma 1.1], but they do not satisfy (a) or (b), so $t(S_\omega \times S_{\omega_1}) > \omega$.

(2) $\Rightarrow$ (1) Since $t(S_\omega \times S_{\omega_1}) > \omega$, $\mathbf{b} = \omega_1$ by Remark 2.4.

Let M be a metric space which is not locally separable, and $Y = S_\omega \oplus M$. Then Y is a k-and $\aleph$-space and $t(Y^2) \leq \omega$, but Y is neither first-countable nor locally separable. Let $f : X \to Y$

be a map. f is an *s-map* if $f^{-1}(y)$ is a separable subset of X for each $y \in Y$. By Theorem 2.6, Remark 2.4 and the fact that every quotient s-image of a locally separable metric space has a point-countable k-network consisting of cosmic subspaces [17; Theorem 2.2], we have the following:

Corollary 2.7 *The following conditions are equivalent*:

(1) $\mathbf{b} = \omega_1$;

(2) *For any pair* (X, Y) *which are quotient s-images of locally separable metric spaces, or k-spaces with a star-countable k-network, then* $t(X \times Y) \leq \omega$ *if and only if*: (*a*) X *or* Y *is first-countable*; *or,* (*b*) *both* X *and* Y *are locally cosmic spaces.*

In view of the proof of Theorem 2.6, it is easy to get the following from Lemma 2.3:

Theorem 2.8 *Let* X *be a k-space with a point-countable k-network consisting of cosmic subspaces. Then* $t(X^2) \leq \omega$ *if and only if* X *is locally cosmic.*

Corollary 2.9 *Let* X *be a quotient s-image of a locally separable metric space, or a k-space with a star-countable k-network. Then* $t(X^2) \leq \omega$ *if and only if* X *is locally cosmic.*

Corollary 2.10 *Let* X *be a k-space with a point-countable k-network. Then* X *has a point-countable base if and only if* $t(X \times S_{\mathbf{b}}) \leq \omega$.

Proof The "if" part follows from Lemmas 2.2 and 2.5. The "only if" part can be obtained from [1].

Let $\mathscr{C}$ be a cover of a space X. X is *dominated* by $\mathscr{C}$ if the union of any $\mathscr{C}' \subset \mathscr{C}$ is closed in X and the union is determined by $\mathscr{C}'$. As is well known, every CW-complex is dominated by a cover of compact metric subsets.

Suppose a space X is dominated by a cover $\mathscr{C}$ of metric subsets. Put $\mathscr{C} = \{C_\alpha : \alpha \in \Gamma\}$. Let $D_0 = C_0$, $D_\beta = C_\beta \setminus \bigcup\{C_\gamma : \gamma < \beta\}$ for each $\beta \in \Gamma$, and let $\mathscr{B}_\beta$ be a σ-locally finite base of the metric subspace D_β. Then $\mathscr{B} = \bigcup\{\mathscr{B}_\beta : \beta \in \Gamma\}$ is a point-countable k-network (in fact, a σ-compact-finite k-network [18, Theorem 3]) of X.

Theorem 2.11 ($\mathbf{b} = \omega_1$) *Let* X *and* Y *be dominated by a cover of metric subsets and* $t(X \times Y) \leq \omega$. *Then one of the following holds*:

(1) X *or* Y *is a metric space*;

(2) *Both* X *and* Y *are* $\aleph$*-spaces.*

Proof If neither X nor Y is a metric space, then both X and Y contain a copy of either S_ω or S_2 by Theorem 13 in [19]. We may assume that X contains a copy of S_ω, so $t(S_\omega \times Y) \leq \omega$; we prove that Y is an $\aleph$-space.

Let Y be dominated by a cover $\mathscr{C}$ of metric subsets. Then Y has a point-countable k-network $\mathscr{B}$. For each $y \in Y$, there exists a countable subfamily $\mathscr{B}' \subset \mathscr{B}$ such that $y \in \mathrm{int}(\cup\mathscr{B}')$. Suppose this is not the case. Then y is not an isolated point. Since Y is a sequential space, Y has countable tightness. There is a countable subset Y_1 of Y such that $y \in \mathrm{cl}Y_1 \setminus Y_1$. Let $\mathscr{P}_1 = \{B \in \mathscr{B} : B \cap Y_1 \neq \emptyset\}$. Then $|\mathscr{P}_1| \leq \omega$, and since $y \notin \mathrm{int}(\cup\mathscr{P}_1)$, then there exists a countable subset Y_2 of Y such that $Y_2 \cap (\cup\mathscr{P}_1) = \emptyset$ and $y \in \mathrm{cl}Y_2 \setminus Y_2$. Let $\mathscr{P}_2 = \{B \in \mathscr{B} : B \cap Y_2 \neq \emptyset\}$. Then $|\mathscr{P}_2| \leq \omega$. Inductively, we obtain a disjoint family $\{Y_\alpha : \alpha < \omega_1\}$ of countable subsets of Y such that:

(a) $y \in \mathrm{cl}Y_\alpha \setminus Y_\alpha$ for each $\alpha < \omega_1$;

(b) For each finite $F_\alpha \subset Y_\alpha$, and a subset $E \subset \omega_1$, $\bigcup\{F_\alpha : \alpha \in E\}$ is closed in Y.

We note that (b) can be obtained from the fact that each element of $\mathscr{B}$ meets at most one Y_α, Y is a sequential space, and $\mathscr{B}$ is a k-network.

Put $A_{\omega_1} = \{y\} \cup \{Y_\alpha : \alpha < \omega_1\}$. Then $t(S_\omega \times A_{\omega_1}) > \omega$ under $\mathbf{b} = \omega_1$ by Lemma 2.3, and $t(S_\omega \times Y) > \omega$, a contradiction.

So there exists a countable subfamily $\mathscr{B}'$ of $\mathscr{B}$ such that $y \in \mathrm{int}(\cup\mathscr{B}')$. For each $B \in \mathscr{B}'$, we choose $C(B) \in \mathscr{C}$ such that $B \subset C(B)$ and $y \in \mathrm{int}(\cup\mathscr{C}')$, where $\mathscr{C}' = \{C(B) : B \in \mathscr{B}'\}$. Since $\mathscr{C}'$ is countable, then $\cup\mathscr{C}'$ is dominated by $\mathscr{C}'$, thus $\cup\mathscr{C}'$ is an $\aleph$-space [20, Proposition 11].

Hence Y is a local $\aleph$-space. Let $\{U_y : y \in Y\}$ be an open cover of Y with each U_y an $\aleph$-space. Since Y is paracompact, there is a locally finite closed refinement $\{F_\delta : \delta \in \Delta\}$ of Y. Let $\mathscr{F}_\delta$ be a σ-locally finite k-network of F_δ. It is easy to check that $\bigcup\{\mathscr{F}_\delta : \delta \in \Delta\}$ is a σ-locally finite k-network of Y. Thus Y is an $\aleph$-space. Similarly, we may show that X is an $\aleph$-space.

Let $\mathbb{I}_\omega$(resp. $\mathbb{I}_{\omega_1}$) be the quotient space obtained from the disjoint union of ω (resp. ω_1) many closed unit intervals $\mathbb{I}$'s by identifying 0 with a point. Then $\mathbb{I}_\omega$ and $\mathbb{I}_{\omega_1}$ are CW-complexes, and $t(\mathbb{I}_\omega \times \mathbb{I}_{\omega_1}) \le \omega$ under $\mathbf{b} > \omega_1$ by Lemma 2.8 in [3], but neither $\mathbb{I}_\omega$ is first-countable nor $\mathbb{I}_{\omega_1}$ is an $\aleph$-space.

3 k-space Property of Product Spaces

In this section, we use Theorem 2.6 to study the k-space property of product spaces. In view of the proof of Lemma 2.4 in [17], we have the following:

Lemma 3.1 *Let $\mathscr{B}$ be a point-countable k-network of a space X. If every first-countable closed subspace of X is locally compact, then $\{P \in \mathscr{B} : \mathrm{cl}P$ is countably compact$\}$ is a k-network of X.*

From Lemmas 3 and 4 in [9], we can get the following:

Lemma 3.2 *If $S_\omega \times X$ is a k-space, then every first-countable closed subspace of X is locally compact.*

By Lemma 3.1 and the property of k-networks, we have:

Lemma 3.3 *Let X be a k-and $\aleph_0$-space. If every first countable closed subspace of X is locally compact, then X is a k_ω-space.*

We say that a space X has a property $(*)$ if every cosmic closed subspace of X is an $\aleph_0$-space. The following spaces have the property $(*)$:

(1) Spaces with a point-countable cs-network [18, Theorem 7].

(2) Spaces with a star-countable k-network, in particular, CW-complexes [21, Proposition 2].

(3) Spaces with a σ-hereditarily closure-preserving k-network, in particular, Lašnev spaces [22].

(4) Fréchet spaces with a point-countable k-network [13, Theorem 5.2].

Theorem 3.4 *The following conditions are equivalent*:

(1) $\mathbf{b} = \omega_1$;

(2) *For any pair (X, Y) which are k-spaces with a point-countable k-network and have the property $(*)$, then $X \times Y$ is a k-space if and only if (X, Y) satisfies the Tanaka condition.*

Proof (2) $\Rightarrow$ (1). Pick $Y = \mathbb{I}(\omega_1) \in \mathscr{L}(\omega_1)$. We know that S_ω and Y are k-spaces with point-countable k-networks and have the property $(*)$, but they do not satisfy one of the (a), (b) and (c) in Tanaka condition, and so $S_\omega \times Y$ is not a k-space. Since S_{ω_1} is a perfect image of Y, $S_\omega \times S_{\omega_1}$ is not a k-space. Thus $\mathbf{b} = \omega_1$ by Theorem 1.4.

(1) $\Rightarrow$ (2). If X, Y satisfy one of (a), (b) and (c) in Tanaka condition, then $X \times Y$ is a k-space.

Let $X \times Y$ be a k-space.

Case 1 *X and Y contain no closed copy of S_ω or S_2.* By Lemma 2.5, both X and Y have a point-countable base.

Case 2 *Only one of X, Y contains a closed copy of S_ω or S_2.* We assume that X contains a closed copy of S_ω or S_2, and Y contains no closed copy of S_ω or S_2. Then $S_\omega \times Y$ is a k-space, and Y is first-countable by Lemma 2.5, so Y is locally compact by Lemma 3.2.

Case 3 *Both X and Y contain a closed copy of S_ω or S_2.* By Lemma 3.2, every first-countable closed subset of X and Y is locally compact, and so X and Y have a point-countable k-network consisting of cosmic subspaces by Lemma 3.1 (Note: A countably compact, k-space with a point-countable k-network is metrizable). By Theorem 2.6, X and Y are locally cosmic spaces, and hence they are local $\aleph_0$-spaces. By Lemma 3.3, they are local k_ω-spaces.

By Theorem 2.8, and Lemma 3.3, we have the following:

Theorem 3.5 *Let X be a k-space with a point-countable k-network and the property $(*)$. Then X^2 is a k-space if and only if X is a locally separable metrizable space or a local k_ω-space.*

Theorems 3.4 and 3.5 may be the fairly general forms of the k-space property of products of spaces with a certain k-network, and they generalize most known results in [9], [10], [11], [19], [22] and [23]. But the authors do not know if the property $(*)$ can be omitted or not.

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