Acta Math. Hungar.
107 (3) (2005), 187–191.

A NOTE ON SEQUENCE-COVERING MAPPINGS

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Abstract. Let $f : X \to Y$ be a mapping. f is called a sequence-covering mapping if in case S is a convergent sequence containing its limit point in Y then there is a compact subset K of X such that $f(K) = S$. It is shown that each quotient and compact mapping of a metric space is sequence-covering.

1. Introduction

In this paper all spaces are assumed to be Hausdorff and maps are continuous and onto. A study of images of metric spaces under certain compact-covering mappings is an important question in general topology [5, 8, 9]. Let $f : X \to Y$ be a mapping. f is called a *compact-covering mapping* [5] if in case L is compact in Y there is a compact subset K of X such that $f(K) = L$. f is called a *compact* (resp. s-)*mapping* if each $f^{-1}(y)$ is compact (resp. separable) in X for each $y \in Y$. Chen [2] had proved that there is a space which is a quotient and compact image of a locally separable metric space and it is not any compact-covering quotient and s-image of a metric space. It is shown that every quotient compact image of a (locally separable) metric space is also a sequence-covering quotient compact image of a (locally separable) metric space (under a different map, in general) [4, 6]; here a mapping $f : X \to Y$ is called *sequence-covering* in the sense of Gruenhage, Michael and Tanaka [5] if in case S is a convergent sequence containing its limit point in Y then there is a compact subset K of X such that $f(K) = S$. The question naturally arises whether every quotient compact mapping of a (locally separable) metric space is sequence-covering [4, 6, 9]. This question is positively answered in this paper.

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*Supported in part by the NSF of China (10271026).

Key words and phrases: sequentially quotient mappings, quotient mappings, sequence-covering mappings.

2000 Mathematics Subject Classification: primary 54E40, 54E20; secondary 54C10, 54D55.

0236–5294/$ 20.00

2. Main results

$f : X \to Y$ is called a *sequentially quotient mapping* [1] if in case $\{y_n\}$ is a convergent sequence in Y then there are a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and a convergent sequence $\{x_i\}$ in X such that each $x_i \in f^{-1}(y_{n_i})$.

LEMMA 2.1 [1]. *Let $f : X \to Y$ be a mapping.*

(1) *If X is a sequential space and f is quotient, then f is sequentially quotient.*

(2) *If Y is a Fréchet space and f is sequentially quotient, then f is pseudo-open.*

THEOREM 2.2 . *Let X be a metric space. If $f : X \to Y$ is a sequentially quotient and compact mapping then f is sequence-covering.*

PROOF. Let a sequence $\{y_n\}$ converge to a point y_0 in Y. We assume without loss of generality that all y_n, y_0 are distinct. Denote $S_1 = \{y_0\} \cup \{y_n : n \in \mathbf{N}\}$, and let $X_1 = f^{-1}(S_1)$, $g = f_{|X_1}$. Then g is a sequentially quotient and compact mapping from a metric space X_1 onto S_1. Since S_1 is a Fréchet space, g is pseudo-open by 2.1. Let $\{U_n : n \in \mathbf{N}\}$ be a decreasing neighborhood base of the compact subset $g^{-1}(y_0)$ in X_1. For each $n \in \mathbf{N}$, $g^{-1}(y_0) \subset U_n$, thus $y_0 \in \operatorname{int}\big(g(U_n)\big)$. Then there is $i_n \in \mathbf{N}$ such that $y_i \in g(U_n)$ as $i \geqq i_n$, so $g^{-1}(y_i) \cap U_n \neq \emptyset$. We may assume that $1 < i_n < i_{n+1}$. For each $j \in \mathbf{N}$, put

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1; \\ f^{-1}(y_j) \cap U_n, & \text{if } i_n \leqq j < i_{n+1}, \end{cases}$$

and $K = g^{-1}(y_0) \cup \{x_j : j \in \mathbf{N}\}$. Since $\{U_n : n \in \mathbf{N}\}$ is a neighborhood base of $g^{-1}(y_0)$ in Y and $x_j \in U_n$ for each $i_n \leqq j < i_{n+1}$, K is compact in X_1 and $g(K) = S_1$, $f(K) = S_1$. Hence f is sequence-covering.

COROLLARY 2.3. *Every quotient and compact mapping of a metric space is sequence-covering.*

PROOF. Let $f : X \to Y$ be a quotient and compact mapping such that X is metric. Then f is a sequentially quotient mapping by 2.1, hence f is a sequence-covering mapping by 2.2.

Let $f : (X, d) \to Y$ be a mapping with d a metric on X. f is a *π-mapping* if for each $y \in Y$ and a neighborhood U of y in Y, $d\big(f^{-1}(y), X \setminus f^{-1}(U)\big) > 0$. Every compact mapping of a metric space is a π-mapping.

EXAMPLE 2.4. There is a quotient and π-mapping $f : (X, d) \to Y$ with d a metric on X such that f is not sequence-covering. Namely, let $Y = \{0\} \cup \{1/n : n \in \mathbf{N}\}$ endowed with the usual subspace topology of the real line $\mathbf{R}$. A collection $\mathcal{D}$ of subsets of $\mathbf{N}$ is said to be *almost disjoint* if $A \cap B$ is finite whenever $A, B \in \mathcal{D}$, $A \neq B$. Using Zorn's Lemma, there exists a collection $\mathcal{A}$ of infinite subsets of $\mathbf{N}$ such that $\mathcal{A}$ is an almost disjoint collection and maximal with respect to these properties. Then $\mathcal{A}$ must be uncountable; denote it by $\{A_\alpha : \alpha \in \Gamma\}$. For each $\alpha \in \Gamma$, put $B_\alpha = \{\alpha\} \cup A_\alpha$, and define a symmetric distance d_α on B_α for each $x, y \in B_\alpha$ as follows:

$$d_\alpha(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1/y, & \text{if } x \neq y \text{ and } x = \alpha; \\ |1/x - 1/y|, & \text{if } x \neq y, x \neq \alpha \text{ and } y \neq \alpha. \end{cases}$$

Then (B_α, d_α) is a metric space. Let $X = \bigoplus_{\alpha\in\Gamma} B_\alpha$, and define a distance d on X for each $x, y \in X$ as follows:

$$d(x,y) = \begin{cases} d_\alpha(x,y), & \text{if } x, y \in B_\alpha \text{ for some } \alpha \in \Gamma; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, d) is a metric space. Define a function $f : X \to Y$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \Gamma; \\ 1/x, & \text{if } x \notin \Gamma. \end{cases}$$

1. *f is continuous.* For each $y \in Y \setminus \{0\}$, $f^{-1}(y) = \oplus\{1/y : 1/y \in A_\alpha\}$ is an open and closed subspace of X. If U is a neighborhood of 0 in Y, then $f^{-1}(U) \cap B_\alpha$ is open in B_α for each $\alpha \in \Gamma$, hence $f^{-1}(U)$ is open in X.

2. *f is quotient.* Let U be a subset of Y with $f^{-1}(U)$ open in X. For each $y \in Y$, if $y \neq 0$, then y is isolated in Y, thus U is a neighborhood of y in Y; if $y = 0$ and U is not a neighborhood of y in Y, then there is an infinite subset I of $\mathbf{N}$ such that $1/n \notin U$ for each $n \in I$. If $I \in \mathcal{A}$, there is $\alpha \in \Gamma$ such that $B_\alpha = \{\alpha\} \cup I$. Since $f^{-1}(U)$ is a neighborhood of α in X, the convergent sequence B_α is eventually in $f^{-1}(U)$, thus the sequence $\{1/n\}_{n\in I}$ is eventually in U, a contradiction. Hence $I \notin \mathcal{A}$, from which there is a $\alpha \in \Gamma$ such that $I \cap A_\alpha$ is infinite by the maximality of $\mathcal{A}$. Then a sequence in $\{1/x : x \in I \cap A_\alpha\}$ is eventually in $f^{-1}(U)$, a contradiction. So U is a neighborhood of 0 in Y. Therefore, f is a quotient mapping.

3. *f is a π-mapping.* If not, there are a $z \in Y$ and a neighborhood U of z in Y such that $d\big(f^{-1}(z), X \setminus f^{-1}(U)\big) = 0$. Then there are sequences $\{z_n\}$

and $\{x_n\}$ in X such that each $z_n \in f^{-1}(z), x_n \in X \setminus f^{-1}(U)$ and $d(z_n, x_n) < 1/n$. Thus each $f(z_n) = z$ and $f(x_n) \notin U$. By the definition of d, there is $\alpha \in \Gamma$ such that $x_n, z_n \in B_\alpha, d(z_n, x_n) < 1/n$, so $\left| f(z_n) - f(x_n) \right| < 1/n$. This implies that the sequence $\left\{ f(x_n) \right\}$ converges to z in Y, a contradiction.

4. *f is not a sequence-covering mapping.* If not, there is a compact subset K of X such that $f(K) = Y$. By the compactness of K, there is a finite subset Γ' of Γ such that $K \subset \bigcup_{\alpha\in\Gamma'} B_\alpha$. Take a $\beta \in \Gamma \setminus \Gamma'$, then A_β is an infinite subset of $\mathbf{N}$ and $A_\beta \cap \left(\bigcup_{\alpha\in\Gamma'} A_\alpha \right)$ is finite, so there is $n_0 \in A_\beta \setminus \left(\bigcup_{\alpha\in\Gamma'} A_\alpha \right) \subset A_\beta \setminus K$. Then there is no $x_0 \in K$ such that $f(x_0) = 1/n_0$, a contradiction. Hence f is not sequence-covering. $\square$

QUESTION 2.5 [9]. Is every quotient π-image of a metric space also a sequence-covering quotient π-image of a metric space?

REMARK 2.6. F. Siwiec [10] defined a *"sequence-covering"* mapping as follows. A map $f : X \to Y$ is called sequence-covering if in case S is a convergent sequence in Y then there is a convergent sequence K of X such that $f(K) = S$. It must be noted that not every quotient and compact mapping is sequence-covering in the sense of Siwiec. For example, let $X = \left(\{0\} \cup \{1/2n : n \in \mathbf{N}\}\right) \oplus \left(\{0\} \cup \{1/2n-1 : n \in \mathbf{N}\}\right), Y = \{0\} \cup \{1/n : n \in \mathbf{N}\}$. X, Y are endowed with the subspace topology of $\mathbf{R}$, and let $f : X \to Y$ be the obvious mapping. Then f is a quotient and compact mapping, and f is not sequence-covering in the sense of Siwiec.

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(Received August 27, 2003)

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