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A NOTE ON SPACES WITH A σ-COMPACT-FINITE WEAK BASE*

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Abstract. In this paper spaces with a σ-compact-finite weak base are discussed, and some characterizations of g-metrizable spaces are obtained by spaces with σ-compact-finite weak base and spaces with a σ-weakly hereditarily closure-preserved weak base.

In this paper all spaces are T_2. Readers may refer to [2] and [6] for unstated definitions.

Let $\mathscr{P}$ be a family of subsets of a space X. $\mathscr{P}$ is called *compact-finite* if any compact subset of X meets at most finitely many members of $\mathscr{P}$; $\mathscr{P}$ is called *closure-preserved* if $\overline{(\bigcup\mathscr{P}')} = \bigcup\{\bar{P} : P \in \mathscr{P}'\}$ for each $\mathscr{P}' \subset \mathscr{P}$; $\mathscr{P}$ is called *hereditarily closure-preserving* if a family $\{H(P) : P \in \mathscr{P}\}$ is closure-preserved for each $H(P) \subset P \in \mathscr{P}$; $\mathscr{P}$ is called *weakly hereditarily closure-preserving* if a family $\{\{p(P)\} : P \in \mathscr{P}\}$ is closure-preserving for each $p(P) \in P \in \mathscr{P}$.

Obviously, a locally finite family for a space is compact-finite and hereditarily closure-preserving, a hereditarily closure-preserving family is closure-preserving and weakly hereditarily closure-preserving. In a k-space, a compact-finite family is a weakly hereditarily closure-preserving family. In certain conditions spaces determined by hereditarily closure-preserving families have some similar properties with spaces determined by compact-finite families.

First, we discuss some properties of weakly hereditarily closure-preserving families. Let $x \in P \subset X$. P is called a *sequential neighborhood* of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x_n : n \geq m\} \subset P$ for some $m \in N$.

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The following Lemmas can be checked directly.

LEMMA 1. *Let $\mathscr{P}$ be a weakly hereditarily closure-preserving family of a space X. If $\mathscr{P}$ is a family of sequential neighborhoods of a point x and there is a non-trivial sequence converging to x in X, then $\mathscr{P}$ is finite.* □

LEMMA 2. *Every point-finite and weakly hereditarily closure-preserving family is compact-finite.* □

LEMMA 3. *Let $\mathscr{P}$ be a weakly hereditarily closure-preserving family of a space X. Put $D = \{x \in X : \mathscr{P}$ is not point-finite at $x\}$. Then $\{P \backslash D : P \in \mathscr{P}\} \cup \{\{x\} : x \in D\}$ is compact-finite.*

PROOF. Since $\{P \backslash D : P \in \mathscr{P}\}$ is a point-finite and weakly hereditarily closure-preserving family of X, it is compact-finite by Lemma 2. If $K \cap D$ is infinite for some compact subset K of X, there are an infinite subset $\{x_i : i \in N\}$ of K and a subset $\{P_i : i \in N\}$ of $\mathscr{P}$ such that each $x_i \in P_i$, thus $\{x_i : i \in N\}$ is closed discrete in K, a contradiction. Therefore, $\{P \backslash D : P \in \mathscr{P}\} \cup \{\{x\} : x \in D\}$ is compact-finite. □

If X is a k-space, then D in Lemma 3 is a closed discrete subset of X.

Let $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$,

(1) $\mathscr{P}_x$ is a *network* of x in X, i.e., $x \in \bigcap \mathscr{P}_x$ and for $x \in U$ with U open in X, $P \subset U$ for some $P \in \mathscr{P}_x$.

(2) If $U, V \in \mathscr{P}_x$, $W \subset U \cap V$ for some $W \in \mathscr{P}_x$.

$\mathscr{P}$ is a *weak base* for X if whenever $G \subset X$ satisfying for each $x \in G$ there is a $P \in \mathscr{P}_x$ with $P \subset G$, then G is open in X. $\mathscr{P}$ is an *sn-network* [7] for X if each member of $\mathscr{P}_x$ is a sequential neighborhood of x in X for each $x \in X$.

$\mathscr{P}_x$ above is called a *wn-network* and an *sn-network* of x, respectively. Every *wn*-network at x is an *sn*-network at x [6, Corollary 1.6.18]. A space X is called a *gf-countable space* if each point of X has a countable *wn*-network. A regular space with a σ-locally finite weak base is called a *g-metrizable space* [10].

Every g-metrizable space is a gf-countable space, every gf-countable space is a sequential space, and every sequential space is a k-space.

For a space X, denote $\mathrm{I} = \{x \in X : x$ is an isolated point of $X\}$.

THEOREM 1. *The following are equivalent for a space X:*

(1) *X has a σ-compact-finite weak base.*

(2) *X is a k-space with a σ-weakly hereditarily closure-preserving weak base.*

(3) *X is a gf-countable space with a σ-weakly hereditarily closure-preserving weak base.*

PROOF. We shall show that (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Let X be a k-space with a σ-weakly hereditarily closure-preserving weak base. X has a σ-compact-finite network by Lemma 3, thus any compact subset of X has a countable network, hence any compact subset of X is metrizable [2, Theorem 3.1.19], and so X is a sequential space. X is gf-countable space by Lemma 1.

Let $\mathscr{P} = \bigcup_{n\in N} \mathscr{P}_n$ be a σ-weakly hereditarily closure-preserving weak base for a gf-countable space X, here each $\mathscr{P}_n$ is a weakly hereditarily closure-preserving family and $\mathscr{P}_n \subset \mathscr{P}_{n+1}$. For each $x \in X$ put $\mathscr{H}_x = \{P \in \mathscr{P} : P$ is a sequential neighborhood of x in $X\}$. If $x \in \mathrm{I}$, then $\{x\}$ is open in X, thus $\{x\} \in \mathscr{P}$, so I is a σ-closed discrete subspace of X. For each $n \in N$, and $P \in \mathscr{P}_n$, put

$$D_n = \{x \in X : \mathscr{P}_n \text{ is not point-finite at } x\},$$

$$W_n(P) = (P\backslash D_n) \cup \{x \in X\backslash \mathrm{I} : P \in \mathscr{H}_x\}.$$

Then $W_n(P) \subset P$. And put $\mathscr{W}_n = \{W_n(P) : P \in \mathscr{P}_n\}$. Then $\mathscr{W}_n$ is point-finite. In fact, for each $x \in X$ we can assume that $x \in X\backslash \mathrm{I}$ by the point-finiteness of the family $\{P\backslash D_n : P \in \mathscr{P}_n\}$, $\mathscr{H}_x \cap \mathscr{P}_n$ is finite by Lemma 1, thus $\mathscr{W}_n$ is point-finite. And $\mathscr{W}_n$ is compact-finite by Lemma 2.

For each $x \in X$, take $\mathscr{B}_x = \{\{x\}\}$ if $x \in \mathrm{I}$, take $\mathscr{B}_x = \{W_n(P) : n \in N, P \in \mathscr{H}_x \cap \mathscr{P}_n\}$ if $x \in X\backslash \mathrm{I}$, we shall show that the subset $\bigcup_{x\in X} \mathscr{B}_x$ of $\bigcup_{n\in N} \mathscr{W}_n \cup \{\{x\} : x \in \mathrm{I}\}$ is a weak base for X. First, for each $x \in X$ and any open neighborhood G of x in X, suppose that $x \in X\backslash \mathrm{I}$, then there are an $n \in N$ and a $P \in \mathscr{H}_x \cap \mathscr{P}_n$ with $P \subset G$, thus $x \in W_n(P) \subset P \subset G$. Secondly, for each $x \in X\backslash \mathrm{I}$, and $U, V \in \mathscr{B}_x$, there are $n, m \in N$ and $P \in \mathscr{H}_x \cap \mathscr{P}_n$, $Q \in \mathscr{H}_x \cap \mathscr{P}_m$ such that $U = W_n(P)$, $V = W_m(Q)$, thus there are a $k \geq \max\{n, m\}$ and $R \in \mathscr{H}_x \cap \mathscr{P}_k$ with $R \subset P \cap Q$, hence $W_k(R) \subset W_n(P) \cap W_m(Q)$. Thirdly, $\mathscr{B}_x$ is an sn-network of x in X. In fact, for each $x \in X\backslash \mathrm{I}$, $n \in N$ and $P \in \mathscr{H}_x \cap \mathscr{P}_n$, let $\{x_i\}$ be a sequence converging to x in X, then $\{x_i\}$ is eventually in P, so $(\{x_i : i \in N\} \cup \{x\}) \cap D_n$ is finite by Lemma 3, hence $\{x_i\}$ is eventually in $(P\backslash D_n) \cup \{x\} \subset W_n(P)$, therefore $W_n(P)$ is a sequential neighborhood of x in X. Thus $\mathscr{B}_x$ is an sn-network of x in X. Suppose that a subset G of X satisfies $B \subset G$ for some $B \in \mathscr{B}_x$ for each $x \in G$, then G is a sequentially neighborhood of each point in G, then G is open in X because X is a sequential space, so $\mathscr{B}_x$ is a wn-network of x in X.

In a word, $\bigcup_{x\in X}\mathscr{B}_x$ is a σ-compact-finite weak base for X. □

The main technique in the proof of Theorem 1 is the $W_n(P)$ constructed, which generate directly a weak base for a space X. The $\mathscr{H}_x$ in proof of Theorem is exactly a *wn*-network $\mathscr{P}_x$ of x in X, it is convenient in proof by using the *sequential neighborhoods* instead of the usual *weak neighborhoods*. Next, we give a direct proof of some properties of g-metrizable spaces by the $W_n(P)$.

COROLLARY 1 [3, 6, 11]. *The following are equivalent for a regular space X:*
(1) *X is a g-metrizable space.*
(2) *X is a k-space with a σ-hereditarily closure-preserving weak base.*
(3) *X is a gf-countable space with a σ-hereditarily closure-preserving weak base.*

PROOF. It only needs to show that (3) ⇒ (1). Let $\mathscr{P}=\bigcup_{n\in N}\mathscr{P}_n$ be a σ-hereditarily closure-preserving weak base for a gf-countable space X, here each $\mathscr{P}_n$ is a family of closed subsets of X by the regularity of X [6, Proposition 2.5.2]. For each $n\in N$ defined D_n, $W_n(P)$ and $\mathscr{W}_n$ as in proof of Theorem 1. To complete the proof, it suffices to show that $\mathscr{W}_n$ is locally finite in X for each $n\in N$ by the proof of Theorem. For each $P\in\mathscr{P}_n$ there is a subset $D_n(P)$ of D_n such that $W_n(P)=(P\backslash D_n)\cup D_n(P)$ because $W_n(P)\subset P\subset(P\backslash D_n)\cup D_n$. For each $x\in X$, if $x\notin D_n$, then $\mathscr{P}_n$ is locally finite at x, thus $\mathscr{W}_n$ is locally finite at x. If $x\in D_n$, there is at most finitely many sets $\{P_i : i\le m_1\}$ of $\mathscr{P}_n$ such that $x\in W_n(P_i)$ for $\mathscr{W}_n$ is point-finite. Let $\{H_k : k\in N\}$ be a decreasing *wn*-network of x in X, there is a $k\in N$ such that at most finitely many members Q_j $(j\le m_2)$ of $\mathscr{P}_n$ with $H_k\cap(Q_j\backslash\{x\})\neq\varnothing$ as $\mathscr{P}_n$ is hereditarily closure-preserving. Let $U=X\backslash(\cup\{P\backslash\{x\} : P\in\mathscr{P}_n\backslash\{Q_j : j\le m_2\}\})\cup(D_n\backslash\{x\})$. If $x\in P\in\mathscr{P}_n\backslash\{Q_j : j\le m_2\}$, then $H_k\cap P=\{x\}$, thus $P\backslash\{x\}$ is closed in X by the closeness of P and the definition of weak bases, and $D_n\backslash\{x\}$ is closed in X by Lemma 3, so U is an open neighborhood of x in X. For each $P\in\mathscr{P}_n$, if $U\cap W_n(P)\neq\varnothing$, then $U\cap(P\backslash D_n)\neq\varnothing$, so $U\cap(P\backslash\{x\})\neq\varnothing$ or $x\in W_n(P)$, therefore $P=Q_j$ for some $j\le m_2$ or $P=P_i$ for some $i\le m_1$, and $\mathscr{W}_n$ is locally finite in X. Consequently, X has a σ-locally finite weak base. □

Y. Tanaka [11] proved that a Lindelöf space with a σ-hereditarily closure-preserving weak base has a countable weak base. The result is true for spaces with a σ-weakly hereditarily closure-preserving weak base.

COROLLARY 2. *Every Lindelöf space with a σ-weakly hereditarily closure-preserving weak base has a countable weak base.*

PROOF. Let $\mathscr{P} = \bigcup_{n\in N} \mathscr{P}_n$ be a σ-weakly hereditarily closure-preserving weak base for a Lindelöf space X, here each $\mathscr{P}_n$ is a weakly hereditarily closure-preserving family of X. First, we shall show that X is a gf-countable space. For each $x \in X\backslash \mathrm{I}$, put $\mathscr{H}_x = \{P \in \mathscr{P} : P$ is a sequential neighborhood of x in $X\}$. If there are an $n \in N$ and an uncountable subset $\{B_\alpha : \alpha < \omega_1\}$ of $\mathscr{H}_x \cap \mathscr{P}_n$, then for each $\alpha < \omega_1$ and any open neighborhood U of x in X, $B_\alpha \cap U \cap (X\backslash\{x\}) \neq \varnothing$ because $X\backslash\{x\}$ is not closed in X. By the induction method, there is a subset $\{x_\alpha : \alpha < \omega_1\}$ of X such that each $x_\alpha \in B_\alpha \cap (X\backslash\{x_\beta : \beta < \alpha\}) \cap (X\backslash\{x\})$, then $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable and closed discrete subspace of X, a contradiction with Lindelöfness of X, thus $\mathscr{H}_x \cap \mathscr{P}_n$ is a countable family for each $n \in N$. Hence X is gf-countable. By Theorem 1, X has a σ-compact-finite weak base. To complete the proof, it is sufficient to show that every compact-finite family is countable in X. Let $\mathscr{Q}$ be any compact-finite family of X, if $\mathscr{Q}$ is not countable, then $\mathscr{Q}$ contains an uncountable subset $\{Q_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$ take a $q_\alpha \in Q_\alpha$, thus $\{q_\alpha : \alpha < \omega_1\}$ is countable because $\mathscr{Q}$ is weakly hereditarily closure-preserving, so q is belong to uncountable many members of $\{Q_\alpha : \alpha < \omega_1\}$ for some $q \in X$, hence $\mathscr{Q}$ is not point-finite, a contradiction. □

Put $S_1 = \{0\} \cup \{1/n : n \in N\}$ with the usual topology. Next, spaces with a σ-compact-finite weak base are characterized by products.

THEOREM 2. *The following are equivalent for a space X:*
(1) *X has a σ-compact-finite base.*
(2) *$X \times S_1$ has a σ-compact-finite weak base.*
(3) *$X \times S_1$ has a σ-weakly hereditarily closure-preserving weak base.*

PROOF. Put $Z = X \times S_1$.

(1) $\Rightarrow$ (2). Suppose that $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$, $\mathscr{Q} = \bigcup_{s\in S_1} \mathscr{Q}_s$ is a σ-compact-finite weak base of the space X and S_1, respectively. For each $z = (x, s) \in Z$, put $\mathscr{H}_z = \{P \times Q : P \in \mathscr{P}_x, Q \in \mathscr{Q}_s\}$, then $\mathscr{H}_z$ is an sn-network of z in Z. Since X is a k-space and S_1 is a locally compact space, Z is a k-space. And any compact subset of Z is metrizable, then Z is a sequential space, thus $\mathscr{H}_z$ is a wn-network of z in Z. Hence $\bigcup_{z\in Z} \mathscr{H}_z$ is a σ-compact-finite weak base of Z.

(2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1). Let $\mathscr{P}$ be a σ-weakly hereditarily closure-

preserving weak base for a space Z. For each $x \in X$, $n \in N$, put $z_n = (x, 1/n)$, then the sequence $\{z_n\}$ converges to $(x,0)$ in Z, thus the family $\{P \in \mathscr{P} : P$ is a sequential neighborhood of $(x,0)$ in $Z\}$ is countable by Lemma 1, so the point $(x,0)$ is gf-countable in Z. Since X is homeomorphic to a closed subspaces $X \times \{0\}$ of Z, X is a gf-countable space with a σ-weakly hereditarily closure-preserving weak base, X has a σ-compact-finite weak base by Theorem 1. □

COROLLARY 3. *The following are equivalent for a regular space* X:
(1) X *is a* g*-metrizable space.*
(2) $X \times S_1$ *has a* σ*-locally-finite weak base.*
(3) $X \times S_1$ *has a* σ*-hereditarily closure-preserving weak base.* □

EXAMPLE. There is a space X with a σ-weakly hereditarily closure-preserving weak base such that X does not any σ-compact-finite weak base or any σ-hereditarily closure-preserving weak base.

Let X be the non-metrizable, paracompact space with a σ-weakly hereditarily closure-preserving base in Example 9 in [1]. Then X has not any σ-hereditarily closure-preserving base by Theorem 5 in [1]. It has been shown that X is not a k-space in [1], thus X has not any σ-compact-finite weak base. By the construction of X, X has a unique non-isolated point $\bar{0}$. If X has a σ-hereditarily closure-preserving weak base $\mathscr{P}$, for each $\bar{0} \in P \in \mathscr{P}$, P is open by the definition of weak base, and for each $x \in X \setminus \{\bar{0}\}$, $\{x\} \in \mathscr{P}$ because $\{x\}$ is open in X, thus X has a σ-hereditarily closure-preserving base, a contradiction. Hence X has not any σ-hereditarily closure-preserving weak base. □

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