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k_ω-SPACES AND Y. TANAKA'S QUESTION*

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Abstract. It is shown that every separable k'-space with a point-countable k-system has a countable k-system, which gives an affirmative answer to Y. Tanaka's Question posed in 1982.

E. Michael established characterizations of paracompact locally compact spaces under quintuple quotient mappings [1]. A. Arhangel'skii introduced k-systems in discussing the relations between metric spaces and quotient mappings [2]. Y. Tanaka investigated spaces with a point-countable k-system, and posed the following questions [3]:

Let X be a space with a point-countable k-system.

QUESTION 1. If X is a k'-space, then does X^2 have a k-system?

QUESTION 2. If X^ω has a k-system, then is X a locally compact space?

QUESTION 3. If X is a separable k'-space with a point-countable k-system, then does X have a countable k-system?

H. Chen positively answered Question 1 in [4], and negatively answered Question 2 in [5]. J. Li and S. Lin partially answered Question 3 in [6]. We give an affirmative answer to the above Question 3.

In this paper, all spaces are Hausdorff spaces. Let us recall some definitions. Let X be a space, and let $\mathcal{P}$ be a cover of X. Then $\mathcal{P}$ is called a k-cover of X if every compact $K \subset X$ is covered by some finite $\mathcal{P}' \subset \mathcal{P}$ (see [6]). A space X is determined by $\mathcal{P}$ if $U \subset X$ is open (closed) in X if and only if $U \bigcap P$ is open (closed) in P for every $P \in \mathcal{P}$. If each element of $\mathcal{P}$ is compact in X, then such a cover is called a k-system according to A. V. Arhangel'skii [2]. A space X is a k-space if it is determined by the cover consisting of all compact subsets of X. A space X is a k'-space if, whenever $x \in \overline{A}$, there exists a compact subset C of X with $x \in \overline{A \bigcap C}$.

A collection $\mathcal{P}$ in X is point-countable if each single point of X meets only countable many members of $\mathcal{P}$.

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Obviously, every k'-space is a k-space.

LEMMA. *Suppose that X is a k'-space, $x \in \overline{A}$, and B is closed in X with $x \notin B$. Then $x \in \overline{A \cap C}$ and $B \cap C = \emptyset$ for some compact subset C of X.*

PROOF. Since X is a k'-space, there exists a compact subset K of X such that $x \in \overline{A \cap K}$. By the regularity of K, $x \in V \subset \overline{V} \subset K \backslash B$ for some open subset V in K, and so $K = \overline{V} \cup (K \backslash V)$. Let $C = \overline{V}$. Then C is compact in X. Thus $x \in \overline{A \cap C}$ and $B \cap C = \emptyset$.

THEOREM. *Suppose that X is a separable k'-space with a point-countable k-system. Then X has a countable k-system.*

PROOF. Let D be a countable dense subset in X, and $\mathcal{K}$ a point-countable k-system for X. Put $\mathcal{P} = \{P \in \mathcal{K} : P \cap D \neq \emptyset\}$. Then $\mathcal{P}$ is countable. Let $\mathcal{P} = \{P_n : n \in N\}$.

(1) $\mathcal{P}$ is a cover of X. For each $x \in X$, $x \in \overline{D}$. Because X is a k'-space, there exists a compact subset C of X such that $x \in \overline{C \cap D}$. For each $y \in C$, put $\{K \in \mathcal{K} : y \in K\} = \big\{K_i(y) : i \in N\big\}$. Suppose that $C \not\subset \cup\mathcal{K}'$ for any finite subset $\mathcal{K}' \subset \mathcal{K}$. Inductively choose $y_n \in C$ such that $y_n \notin K_i(y_j)$ for $i, j < n$. So every $K_i(y_j)$ contains only finitely many y_n's. Let $Y = \{y_n : n \in N\}$. Since C is compact in X, Y has an accumulation point z in C. Put $Z = Y \backslash \{z\}$. Then Z is not closed in X, and so there is $K \in \mathcal{K}$ such that $K \cap Z$ is not closed in X, and hence K contains infinitely many y_n's. Thus $K = K_i(y_j)$ for some $i, j \in N$, a contradiction. Therefore, there exists a finite subset $\mathcal{K}' \subset \mathcal{K}$ such that $C \subset \cup\mathcal{K}'$, and so $x \in \overline{K \cap D}$ for some $K \in \mathcal{K}'$. Thus $x \in K \in \mathcal{P}$. Hence $\mathcal{P}$ is a cover of X.

(2) If $A \subset D$ and $x \in \overline{A}$, then there exists $m \in N$ such that $x \in \overline{A \cap P_m}$. Since X is a k'-space, there exists a compact subset C of X such that $x \in \overline{A \cap C}$. By (1), there is a finite subset $\mathcal{K}' \subset \mathcal{K}$ such that $C \subset \cup\mathcal{K}'$, and so $x \in \overline{A \cap P}$ for some $P \in \mathcal{K}'$. Thus $P \cap D \neq \emptyset$, and $P \in \mathcal{P}$. Hence $x \in \overline{A \cap P_m}$ for some $m \in N$.

(3) $\mathcal{P}$ is a k-cover of X. Suppose not, then there is a compact subset K of X such that $K \not\subset \cup\mathcal{P}'$ for any finite subset $\mathcal{P}' \subset \mathcal{P}$. Since $\mathcal{P}$ is a cover of X, we can inductively choose $x_i \in K \cap P_{n_i} \backslash \cup_{j<n_i} P_j$, where $n_1 = 1$ and $n_i < n_{i+1}$. Because K is compact in X, $\{x_i : i \in N\}$ has an accumulation point in K. Without loss of generality, let $\{x_i : i \in N\}$ have an accumulation point y and $x_i \neq y$ for every $i \in N$. Because $x_i \in \overline{D}$ for every $i \in N$, by Lemma, we can inductively choose a compact subset C_i of X such that $x_i \in \overline{D \cap C_i}$ and $C_i \cap \big(\{y\} \cup (\cup_{j<n_i} P_j)\big) = \emptyset$, so $y \in \overline{D \cap (\cup_{i \in N} C_i)}$. By (2), $y \in \overline{D \cap (\cup_{i \in N} C_i) \cap P_m}$ for some $m \in N$. Taking $i_0 \in N$ with $n_{i_0} \leqq m < n_{i_0+1}$, then when $i > i_0$, $P_m \cap C_i = \emptyset$. Thus $y \in \overline{D \cap C_i \cap P_m} \subset C_i \subset X \backslash \{y\}$ for some $i \leqq i_0$, a contradiction. Hence $\mathcal{P}$ is a k-cover of X.

(4) $\mathcal{P}$ is a countable k-system for X. Suppose that there is $F \subset X$ such that $F \cap P$ is closed in X for each $P \in \mathcal{P}$, but F is not closed in X. Since X is a k-space, $F \cap C$ is not closed in X for a suitable compact C, and so, by (2), $C \subset \cup\mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$. However, $F \cap C = \cup\{(F \cap P) \cap C : P \in \mathcal{P}'\}$ is closed in X, a contradiction. Hence X is determined by $\mathcal{P}$, and $\mathcal{P}$ is a k-system for X.

References

[1] E. Michael, A quintuple quotient quest, *Gen. Top. Appl.,* **2** (1972), 91–138.

[2] A. Arhangel'skii, Quotient mappings of metric space, *Dokl. Akad. Nauk SSSR,* **155** (1964), 247–250 (in Russian).

[3] Y. Tanaka, Point-countable k-systems and products of k-spaces, *Pacific J. Math.,* **141** (1982), 199–208.

[4] H. Chen, The products of k'-spaces with a point-countable k-system, *Chinese Bull. of Science,* **31** (1986), 79–79.

[5] H. Chen, An answer to a conjecture on the countable products of k-spaces, *Proc. Amer. Math. Soc.,* **123** (1995), 583–587.

[6] J. Li and S. Lin, Spaces with compact-countable k-systems, *Acta Math. Hungar.,* **93** (2001), 1–6.

[7] E. Michael, Bi-quotient maps and cartesian products of quotient maps, *Ann. Inst. Four (Grenoble),* **18** (1968), 287–302.

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