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GENERAL TOPOLOGY

Regular Covers and Metrization

by

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Summary. In this paper it is showed that a regular and k-space with a regular k-network is metrizable, which generalized related results of A. Archangielskiĭ, H. W. Martin, M. Sakai, K. Tamano and Y. Yajima.

In 1960, A. Archangielskiĭ [3] proved that a space with a regular base is metrizable. In 1976, H. W. Martin [9] proved that a space with a regular weak base is metrizable. In 1998, M. Sakai, K. Tamano and Y. Yajima [11] proved that a regular and Fréchet space with a regular k-network is metrizable. In this paper we show that a regular and k-space with a regular k-network is metrizable, which generalizes related results of [4], [9] and [11].

Recall some related concepts. In this paper all spaces are T_2. $\tau(X)$ denotes a topology of a space X.

DEFINITION 1 [5]. Let X be a space, and $P \subset X$.

(1) A sequence $\{x_n\}$ in X is called eventually in P, if the $\{x_n\}$ converges to x, and there is $m \in N$ such that $\{x\} \cup \{x_n : n \geqslant m\} \subset P$.

(2) P is called a sequential neighbourhood of x in X, if whenever a sequence $\{x_n\}$ converges to x in X, then $\{x_n\}$ is eventually in P.

(3) X is called a sequential space, if every $A \subset X$ which is a sequential neighbourhood of each of its points is open in X.

(4) X is called a k-space, if for every $A \subset X$ such that $K \cap A$ is closed in K for each compact K in X, A is closed in X.

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(5) X is called a Fréchet space, if for each $x \in \mathrm{cl}(A) \subset X$ there is a sequence $\{x_n\}$ in A which converges to x in X.

Every Fréchet space is a sequential space. Every sequential space is a k-space.

DEFINITION 2. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a cs^*-network for X [6], if for every sequence $\{x_n\}$ converging to x and a neighbourhood V of x in X there is $P \in \mathcal{P}$ such that some subsequence of $\{x_n\}$ is eventually in P and $P \subset V$.

(2) $\mathcal{P}$ is called a k-network for X [10], if for every $K \subset V$ with K compact and V open in X there is a finite subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset V$.

Every k-network by closed subsets is a cs^*-network for a space. From Corollary 3.4 in [7] a k-space with a point-countable k-network is a sequential space.

DEFINITION 3 [1]. Let $\mathcal{P}$ be a cover for a space X.

(1) $\mathcal{P}$ is called a point-regular cover for X, if for every $x \in U \in \tau(X)$ $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite.

(2) $\mathcal{P}$ is called a regular cover for X, if for every $x \in U \in \tau(X)$ there is an open neighbourhood V of x in X such that $\{P \in (\mathcal{P})_V : P \not\subset U\}$ is finite.

$\mathcal{P}$ is called a point-regular (or regular) cs^*-network (or k-network) for X if $\mathcal{P}$ is a point-regular (or regular) cover and a cs^*-network (or k-network) for X.

Every regular cover is a point-regular cover for a space X.

LEMMA 4 [8]. *If $\mathcal{P}$ is a regular cover for a regular space X, then $\{\overline{P} : P \in \mathcal{P}\}$ also is a regular cover for X.*

LEMMA 5. *Let $\mathcal{P}$ be a cover for a space X. $\mathcal{P}$ is point-regular if and only if for each $x \in X$, if $\{P_n : n \in N\}$ is an infinite subset of $(\mathcal{P})_x$ and U is a sequential neighbourhood of x in X, then there is $m \in N$ such that $P_n \subset U$ for each $n > m$.*

Proof. Let $\mathcal{P}$ be a point-regular cover for a space X. Suppose that $x \in X$, $\{P_n : n \in N\}$ is an infinite subset of $(\mathcal{P})_x$, and U is a sequential neighbourhood of x in X. If there is no $m \in N$ such that $P_n \subset U$ for each $n > m$, then there is an infinite subset $\{P_{n_k} : k \in N\}$ of $\{P_n : n \in N\}$ such that each $P_{n_k} \not\subset U$. Take $x_k \in P_{n_k} \setminus U$ for each $k \in N$, then the sequence $\{x_k\}$ converges to $x \in U$ because $\mathcal{P}$ is a point-regular cover for

X. This is a contradiction because P is a sequential neighbourhood of x in X. Conversely, if $\mathcal{P}$ is not point-regular for X, then there are a point $x \in U \in \tau(X)$ and an infinite subset $\{P_n : n \in N\}$ of $\{P \in (\mathcal{P})_x : P \not\subset U\}$, thus there is not any $m \in N$ such that $P_n \subset U$ for each $n > m$. □

Let $T_0 = \{a_n\}$ be a sequence converging to $x_0 \notin T_0$ and let each $T_n (n \in N)$ be a sequence converging to $a_n \notin T_n$. Let $T = \bigoplus_{n \in N}(T_n \cup \{a_n\})$. $S_2 = \{x_0\} \cup (\bigcup_{n \in w} T_n)$ is a quotient space obtained from the topological sum $(T_0 \bigcup \{x_0\}) \bigoplus T$ by identifying each $a_n \in T_0$ with $a_n \in T$. S_2 is also called a Arens space [2].

THEOREM 6. *Let X be a sequential space with a point-regular cs*-network. If X is not a Fréchet space, then X contains a closed copy of S_2.*

Proof. Let $\mathcal{P}$ be a point-regular cs^*-network for a space X with a topology τ. First, we show that $\mathcal{P}$ is point-countable. If there is a point $x \in X$ such that $(\mathcal{P})_x$ is uncountable, by the point-regularity of $\mathcal{P}$ for each $y \neq x$, $\{P \in (\mathcal{P})_x : y \in P\}$ is finite, thus there are an infinite subset $\{P_n : n \in N\}$ of $(\mathcal{P})_x, x_n \in P_n \setminus \{x\}$ and $k \in N$ such that each $\{x_n\}$ belongs exactly to k elements of $(\mathcal{P})_x$, i.e. $\mathrm{ord}(x_n, (\mathcal{P})_x) = k$ for each $n \in N$. By Lemma 5, the sequence $\{x_n\}$ converges to x. Since $\mathcal{P}$ is a cs^*-network for X, there are a subset $\{F_i : i \in N\}$ of $(\mathcal{P})_x$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_j} : j \geqslant i\} \subset F_i \subset X \setminus \{x_{n_j} : j < i\}$ for each $i \in N$, thus $\mathrm{ord}(x_{n_i}, (\mathcal{P})_x) \geqslant i$, a contradiction. Hence $\mathcal{P}$ is point-countable.

For a subset A of X, denote $\mathrm{cl}_s(A) = \{x \in X$: there is a sequence in A converging to x in $X\}$.

If X is not a Fréchet space, then there is a subset H of X with $\mathrm{cl}_s(H) \neq \overline{H}$. Since X is a sequential space, there is a sequence $\{x_n\}$ in $\mathrm{cl}_s(H)$ converging to $x \in X \setminus \mathrm{cl}_s(H)$ in X. We can assume that all x_n's are distinct and each $x_n \notin H$. Since X is a T_2-space, there is a sequence $\{V_n\}$ of pairwise disjoint open subsets in X with each $x_n \in V_n$. For each $n \in N$, there is a sequence $\{x_{nm}\}$ in $H \cap V_n$ converging to x_n in X. Put $C = \{x\} \cup \{x_n : n \in N\} \cup \{x_{nm} : n, m \in N\}$, and define a topology on C as follows: U is open in C if and only if U is sequentially open in $(C, \tau|_C)$. The set C endowed with the above topology is denoted by σC. We shall show that σC is homeomorphic to S_2.

Since σC is a sequential space, its topology is defined by convergent sequences. If σC is not homeomorphic to S_2, then there is a convergent sequence $\{y_k\}$ in σC such that $\{y_k\}$ converges to a point y in σC and the sequence $\{y_k\}$ meets infinite many of the sequences $\{x_{nm}\}_{m \in N}$. Since $x \notin \mathrm{cl}_s(H)$, $y \neq x$, thus $y \in V_i$ for some $i \in N$, hence there is $j \in N$ such that $y_k \in V_i$ for each $k \geqslant j$, a contradiction because the elements of $\{V_n\}$ are disjoint. So σC is homeomorphic to S_2.

Put $K = \{x\} \cup \{x_n : n \in N\}$, $\mathcal{R} = \{P \in \mathcal{P} : P \cap \{x_{nm} : n, m \in N\} \neq \emptyset$, and $\overline{P} \cap K = \emptyset\}$. Then $\mathcal{R}$ is countable. Let $\mathcal{R} = \{P_k : k \in N\}$. For each $n \in N$, there is $m_n \in N$ such that $\{x_{nm} : m \geqslant m_n\} \subset X \setminus \bigcup_{k \leqslant n} \overline{P_k}$. Take $S = K \bigcup \{x_{nm} : n \in N, m \geqslant m_n\}$, then σS is still homeomorphic to S_2. If S is not closed in X, there is a sequence $\{x_{n_i m_i}\}$ in S converging to $x' \notin S$. We can assume that each $n_{i+1} > n_i$. Put $K_1 = \{x'\} \cup \{x_{n_i m_i} : i \in N\}$. Then $K_1 \cap K = \emptyset$, thus there is an open subset U in X such that $K_1 \subset U \subset \overline{U} \subset X \setminus K$. Since $\mathcal{P}$ is a cs^*-network for X, there is $P \in \mathcal{P}$ such that some subsequence of $\{x_{n_i m_i}\}$ is eventually in P and $P \subset U$, hence $P = P_j$ for some $j \in N$, and $x_{n_i m_i} \notin P$ for each $n_i \geqslant j$, a contradiction. Hence S is closed in X. Since X is a sequential space, S is a sequential space, thus $\sigma S = S$. Therefore X contains a closed copy of S_2. □

LEMMA 7. *The space S_2 has not any regular cs*-network.*

Proof. Represent the space S_2 as $\{x_0\} \cup \{x_{nm} : n \in N, m \in \omega\}$, where the sequence $\{x_{n0}\}$ converges to x_0, and the sequence $\{x_{nm}\}$ converges to x_{n0} for each $n \in N$. Let $\mathcal{P}$ be a cs^*-network for S_2. Since $\{x_{nm} : m \in \omega\}$ is open in S_2 for each $n \in N$, there are $P_n \in \mathcal{P}$ and $m_n \in N$ such that $\{x_{n0}, x_{nm_n}\} \subset P_n$ and the P_n' s are disjoint. Put $U = S_2 \setminus \{x_{nm_n} : n \in N\}$. Then U is an open neighbourhood of x_0 in S_2. For each open neighbourhood V of x_0 in S_2, there is $k \in N$ such that $x_{n0} \in V$ for each $n > k$, thus $P_n \cap V \neq \emptyset$ and $P_n \not\subset U$. Hence $\mathcal{P}$ is not a regular cover for S_2, so S_2 has not any regular cs^*-network. □

LEMMA 8 [11]. *Every regular and Fréchet space with a regular k-network is metrizable.*

In this paper the main result is that

THEOREM 9. *Every regular and k-space with a regular k-network is metrizable.*

Proof. Let X be a regular and k-space with a regular k-network. By Lemma 4, X has a regular k-network $\mathcal{P}$ by closed subsets, then $\mathcal{P}$ is a cs^*-network for X. By the proof of Theorem 6, $\mathcal{P}$ is point-countable, thus X is a sequential space (cf. Corollary 3.4 in [7]), so X is a metrizable space by Theorem 6, Lemma 7 and Lemma 8.

DEFINITION 10 [4]. Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a family of subsets of a space X, which satisfies that

(1) For each $x \in X, \mathcal{P}_x \subset (\mathcal{P})_x$ and if $x \in G \in \tau(X)$, then $P \subset G$ for some $P \in \mathcal{P}_x$;

(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a weak base for X, if for every $G \subset X$ such that for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G, G$ is open in X.

COROLLARY 11 [9]. *Every space with a regular weak base is metrizable.*

Proof. By Theorem 9, it need only to show that every space with a regular weak base is a regular and sequential space with a regular cs^*-network.

Suppose $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ is a regular weak base for a space X. First, for every $x \in X$ and $P \in \mathcal{P}_x$, P is a sequential neighbourhood of x in X. If not, there is a sequence $\{x_n\}$ in $X \backslash P$ converging to x. Let $U = X \backslash \{x_n : n \in N\}$, U is not open in X. But, for each $z \in U$ there is $Q \in \mathcal{P}_z$ such that $Q \subset U$, U is open in X by Definition 10, a contradiction.

(1) $\mathcal{P}$ is a cs^*-network for X. If a sequence $\{x_n\}$ converges to $x \in V$ with V open in X, there is $P \in \mathcal{P}_x$ such that $P \subset V$. Since P is a sequential neighbourhood of x in X, some subsequence of $\{x_n\}$ is eventually in P.

(2) X is a sequential space. If U is sequentially open in X, for each $x \in U, P \subset U$ for some $P \in \mathcal{P}_x$ by Lemma 5, thus U is open in X.

(3) X is a regular space. By Theorem 6 and Lemma 7, X is a Fréchet space. If $x \in U$ with U open in X, there is an open neighbourhood V of x in X such that $V \subset U$ and $\{P \in (\mathcal{P})_V : P \not\subset U\}$ is finite. Let $z \in \overline{V}$, suppose z is not an isolated point in X, there is a sequence $\{z_k\}$ in $V \backslash \{z\}$ converging to z, there is a subset $\{P_n : n \in N\}$ of $\mathcal{P}_z$ such that each $P_n \subset X \backslash \{z_i : i \leqslant n\}$. Since each P_n is a sequential neighbourhood of z in X, sequence $\{z_k\}$ is eventually in P_n, $\{P_n : n \in N\}$ is an infinite set, thus $z \in P_n \subset U$ for some $n \in N$. Hence $\overline{V} \subset U$.

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