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# Sequence-covering maps of metric spaces

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**Abstract**

Let $f:X\to Y$ be a map. $f$ is a sequence-covering map if whenever $\{y_n\}$ is a convergent sequence in $Y$, there is a convergent sequence $\{x_n\}$ in $X$ with each $x_n\in f^{-1}(y_n)$. $f$ is a 1-sequence-covering map if for each $y\in Y$, there is $x\in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$ in $Y$ there is a sequence $\{x_n\}$ converging to $x$ in $X$ with each $x_n\in f^{-1}(y_n)$. In this paper we investigate the structure of sequence-covering images of metric spaces, the main results are that

(1) every sequence-covering, quotient and s-image of a locally separable metric space is a local $\aleph_0$-space;
(2) every sequence-covering and compact map of a metric space is a 1-sequence-covering map.


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## 1. Introduction

"Mappings and spaces" is one of the questions in general topology [1]. Spaces with certain k-networks play an important role in the theory of generalized metric spaces [7,8]. In the past the relations among spaces with certain k-networks were established by means of maps [1,11], in which quotient maps, closed maps, open maps and compact-covering maps were powerful tool. In recent years, sequence-covering maps introduced by Siwiec in [25] cause attention once again [12,14,16,17,19,30,31,33,36]. Partly, that is because

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sequence-covering maps are closely related to the question about compact-covering and s-images of metric spaces [22,23], certain quotient images of metric spaces [29,30], and they are a suitable map associated metric spaces with spaces having certain cs-networks [14, 27]. The present paper contributes to the problem of characterizing the certain quotient images or sequence-covering images of metric spaces, which is inspired by the following questions.

**Question 1.1** [33]. What is a nice characterization for a quotient and s-image of a locally separable metric space?

**Question 1.2** [32]. For a sequential space $X$ with a point-regular cs-network, characterize $X$ by means of a nice image of a metric space.

Those questions are motivated by the following assertions:

(1) A space is a quotient and s-image of a metric space if and only if it is a sequential space with a point-countable cs*-network [29];

(2) A space is an open and compact image of a metric space if and only if it has a point-regular base [1].

First, recall some basic definitions. All spaces are considered to be regular and $T_1$, and all maps continuous and onto.

**Definition 1.3.** Let $f: X \to Y$ be a map.

(1) $f$ is an *s-map* if each $f^{-1}(y)$ is separable.

(2) $f$ is a *compact map* if each $f^{-1}(y)$ is compact.

(3) $f$ is a *compact-covering map* [21] if each compact subset of $Y$ is the image of some compact subset of $X$.

(4) $f$ is a *sequence-covering map* [25] if each convergent sequence of $Y$ is the image of some convergent sequence of $X$.

(5) $f$ is a *sequentially quotient map* [2] if for each convergent sequence $L$ of $Y$, there is a convergent sequence $S$ of $X$ such that $f(S)$ is a subsequence of $L$.

**Definition 1.4.** Let $X$ be a space, and let $\mathcal{P}$ be a cover of $X$.

(1) $\mathcal{P}$ is a *network* if whenever $x \in U$ with $U$ open in $X$, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is a *k-network* [24] if whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

(3) $\mathcal{P}$ is a *cs-network* [10] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with $U$ open in $X$, then $\{x\} \cup \{x_n\colon n \geqslant m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is a *cs*-network* [6] if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with $U$ open in $X$, then $\{x\} \cup \{x_{n_i}\colon i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

A space is a *cosmic space* [21] if it has a countable network. A space is an $\aleph_0$-*space* [21] if it has a countable k-network, and it is equivalent to a space with a countable cs-network or a countable cs*-network.

**Definition 1.5** [5]. Let $X$ be a space.

(1) Let $x \in P \subset X$. $P$ is a *sequential neighborhood* of $x$ in $X$ if whenever $\{x_n\}$ is a sequence converging to the point $x$, then $\{x_n\colon n \geqslant m\} \subset P$ for some $m \in \mathbb{N}$.
(2) Let $P \subset X$. $P$ is a *sequentially open subset* in $X$ if $P$ is a sequential neighborhood of $x$ in $X$ for each $x \in P$.
(3) $X$ is a *sequential space* if each sequentially open subset in $X$ is open.

We recall that a cover $\mathcal{P}$ is *point-countable* if $\{P \in \mathcal{P}\colon x \in P\}$ is countable for each $x \in X$, $\mathcal{P}$ is *star-countable* if $\{Q \in \mathcal{P}\colon Q \cap P \neq \emptyset\}$ is countable for each $P \in \mathcal{P}$, $\mathcal{P}$ is *locally countable* if for each $x \in X$, there is an open neighborhood $V$ of $x$ in $X$ such that $\{P \in \mathcal{P}\colon P \cap V \neq \emptyset\}$ is countable. A space $X$ is *metalindelöf* if each open cover of $X$ has a point-countable open refinement.

## 2. Sequential separability

**Definition 2.1** [28,37]. A space $X$ is *sequentially separable* if $X$ has a countable subset $D$ such that for each $x \in X$, there is a sequence $\{x_n\}$ in $D$ with $x_n \to x$. $D$ is called a sequentially dense subset of $X$.

Liu and Tanaka [18] showed that every cosmic space with a point-countable cs-network is an $\aleph_0$-space, in which key step is that every cosmic space is sequentially separable. Michael [21] proved that a space $X$ is a cosmic space if and only if it is an image of a separable metric space. We shall show that every sequentially separable space has a similar result. Recall some basic definitions. A space $X$ is *Fréchet* [5], if whenever $x \in \mathrm{cl}(A) \subset X$, there is a sequence in $A$ converging to the point $x$. A space $X$ is *developable* [4] if $X$ has a development, i.e., there is a sequence $\{\mathcal{U}_n\}$ of open covers of $X$ such that $\{\mathrm{st}(x, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is a local base of $x$ for each $x \in X$. It is clear that

developable spaces $\Rightarrow$ first countable spaces
$\Rightarrow$ Fréchet spaces $\Rightarrow$ sequential spaces.

**Lemma 2.2.** *Sequential separability is preserved by a map.*

**Lemma 2.3.** *Every separable and Fréchet space is sequentially separable.*

**Lemma 2.4.** *Let $X$ be a sequentially separable space.*

(1) *If $X$ has a point-countable cs*-network, $X$ is a cosmic space.*
(2) *If $X$ has a point-countable k-network, $X$ is a cosmic space.*
(3) *If $X$ has a point-countable cs-network, $X$ is an $\aleph_0$-space.*
(4) *If $X$ has a star-countable k-network, $X$ is an $\aleph_0$-space.*

**Proof.** Let $X$ be a sequentially separable space with a countable and sequentially dense subset $D$. If $X$ has a point-countable cs*-network $\mathcal{P}$, put

$$\mathcal{P}' = \{P \in \mathcal{P}\colon\ P \cap D \neq \emptyset\}.$$

Then $\mathcal{P}'$ is countable. For each $x \in U$ with $U$ open in $X$, there is a sequence $\{x_n\}$ in $D$ with $x_n \to x$. Since $\mathcal{P}$ is a cs*-network for $X$, $\{x\} \cup \{x_{n_i}\colon i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$, and $P \in \mathcal{P}'$. Thus $\mathcal{P}'$ is a countable network of $X$, and $X$ is a cosmic space. If $X$ has a point-countable k-network $\mathcal{P}$, put

$$\mathcal{P}' = \{\mathrm{cl}(P)\colon\ P \in \mathcal{P},\ P \cap D \neq \emptyset\}.$$

Then $\mathcal{P}'$ is countable. For each $x \in U$ with $U$ open in $X$, there are an open set $V$ of $X$ and a sequence $\{x_n\}$ in $D$ such that $x_n \to x \in V \subset \mathrm{cl}(V) \subset U$. Since $\mathcal{P}$ is a k-network for $X$, $\{x_{n_i}\colon i \in \mathbb{N}\} \subset P \subset V$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$, and $\mathrm{cl}(P) \in \mathcal{P}'$ and $x \in \mathrm{cl}(P) \subset U$. Thus $\mathcal{P}'$ is a countable network of $X$, and $X$ is a cosmic space. (3) has been proved by Liu and Tanaka in [18]. Since a cosmic space with a star-countable k-network is an $\aleph_0$-space [19, Proposition 2], (4) is also true by (2). □

**Theorem 2.5.** *The following are equivalent for a space $X$:*

(1) *$X$ is a sequentially separable space.*
(2) *$X$ is an image of a separable and first countable space.*
(3) *$X$ is an image of a separable and developable space.*

**Proof.** We only need to show that (1) ⇒ (3). Let $X$ be a sequentially separable space, and let $D = \{d_n\colon n \in \mathbb{N}\}$ be a sequentially dense subset of $X$. For each $x \in X$, take a fixed $S_x = \{d_{x,n}\colon n \in \mathbb{N}\} \subset D$ with $S_x \to x$. Suppose that each $d_{x,n} = x$ if $x \in D$ and each $d_{x,n}$ is distinct if $x \in X \setminus D$. Topology of $X$ is denoted by $\tau$. A new topology $\tau^*$ of $X$ is defined as follows: for each $x \in U \subset X$, $U$ is a neighborhood of $x$ in $(X, \tau^*)$ if and only if $\{d_{x,n}\colon n \geqslant m\} \subset U$ for some $m \in \mathbb{N}$. Then $\tau^*$ is a topology of $X$.

(a) $\tau^*$ is separable, locally compact and $T_2$.

$\{x\} \cup \{d_{x,n}\colon n \geqslant m\}$ is a compact neighborhood of $x$ in $(X, \tau^*)$ for each $x \in X$ and each $m \in \mathbb{N}$.

(b) $\tau^*$ is developable.

We assume that $X \setminus D$ is uncountable and $\bigcup\{S_x\colon x \in X \setminus D\} = D$. For each $n \in \mathbb{N}$, put $F_n = \{d_i\colon i \leqslant n\}$,

$$\mathcal{U}_n = \{\{x\} \cup (S_x \setminus F_n)\colon x \in X \setminus D\} \cup \{\{x\}\colon x \in F_n\}.$$

Then $\mathcal{U}_n$ is an open cover of $X$ and for each $x \in X$,

$$\mathrm{st}(x, \mathcal{U}_n) = \begin{cases} \{X\} \cup (S_x \setminus F_n), & x \in X \setminus D, \\ \{x\}, & x \in F_n. \end{cases}$$

$\{\mathrm{st}(x, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is a local base of $x$ in $(X, \tau^*)$. So $\{\mathcal{U}_n\}$ is a development in $(X, \tau^*)$, and $(X, \tau^*)$ is a developable space.

Since $\tau \subset \tau^*$, the identical map $\mathrm{id}_x : (X, \tau^*) \to (X, \tau)$ is continuous, and $X$ is an image of a separable and developable space. □

**Corollary 2.6.** *Every cosmic space is sequentially separable.*

**Remark 2.7.**

(1) A separable and sequential space $\not\Rightarrow$ sequentially separable; see Example 9.3 in [9] or Example 2.8.16 in [11].

(2) A $T_2$, sequential and cosmic space with a point-countable cs*-network $\not\Rightarrow$ an $\aleph_0$-space; see Example 4 in [17].

(3) A $T_2$, first countable, separable space with a locally countable k-network $\not\Rightarrow$ a space with a point-countable cs*-network; see Half-Disc Topology in [26].

(4) Every separable and Fréchet space with a point-countable k-network is an $\aleph_0$-space [9].

## 3. A sequence-covering and s-image of a locally separable metric space

Find a simple internal characterization of a quotient and s-image of a locally separable metric space is still an unsolved question [15,33]. By Lemma 2.4(3), Tanaka and Xia [33] showed that a space is a sequence-covering and s-image of a locally separable metric space if and only if it has a point-countable cs-network consisting of cosmic subspaces. On the other hand, Velichko [34] posed an interesting question about quotient and s-images of metric spaces as follows: Find a $\Phi$-property such that a space $Y$ is a quotient and s-image of a metric and $\Phi$-space if and only if $Y$ is a $\Phi$-space which is a quotient and s-image of a metric space. Velichko [34] proved that a space $Y$ is a pseudo-open and s-image of a locally separable metric space if and only if $Y$ is a locally separable space which is a pseudo-open and s-image of a metric space. In this section, we shall show that a local $\aleph_0$-property is a positive answer about Velichko's question if the quotient map is replaced by a sequence-covering map.

**Definition 3.1.** Let $X$ be a space, and let $\mathcal{P}$ be a cover for $X$.

(1) $\mathcal{P}$ is an *sn-cover* (i.e., *sequential neighborhood cover*) if each element of $\mathcal{P}$ is a sequential neighborhood of some point in $X$, and for each $x \in X$, some $P \in \mathcal{P}$ is a sequential neighborhood of $x$.

(2) $\mathcal{P}$ is an *so-cover* (i.e., *sequentially open cover*) if each element of $\mathcal{P}$ is sequentially open in $X$.

**Lemma 3.2.** *Let $\mathcal{P}$ be a point-countable cs-network of a space $X$ which is closed under finite intersections, and let $\mathcal{U}$ be an sn-cover for $X$. Put*

$$\mathcal{P}' = \{P \in \mathcal{P}\colon\ P \subset U \text{ for some } U \in \mathcal{U}\}.$$

*Then $\mathcal{P}'$ is still a cs-network for $X$.*

**Proof.** Let $x \in W$ with $W$ open in $X$. If $\{x_n\}$ is a sequence converging to the point $x$ in $X$, put

$$\mathcal{P}_x = \{P \in \mathcal{P}\colon\ x \in P \subset W \text{ and } P \text{ contains all but finitely many } x_n\} = \{P_n\colon\ n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, take $Q_n = \bigcap_{i \leqslant n} P_i$, then $Q_n \in \mathcal{P}_x$. Let $U_x \in \mathcal{U}$ be a sequential neighborhood of $x$ in $X$. If there is $q_n \in Q_n \setminus U_x$ for each $n \in \mathbb{N}$, suppose that $G$ is open in $X$ with $x \in G$, then $P_k \subset G$ for some $k \in \mathbb{N}$ because $\mathcal{P}$ is a cs-network for $X$, thus $q_n \in Q_n \subset P_k \subset G$ when $n \geqslant k$, and $q_n \to x$, a contradiction. Hence $Q_m \subset U_x$ for some $m \in \mathbb{N}$, and $Q_m \in \mathcal{P}'$. Therefore, $\mathcal{P}'$ is a cs-network for $X$. □

The point-countability of $\mathcal{P}$ in Lemma 3.2 is essential. Let $X = \mathbb{N} \cup \{p\}$, here $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $\mathcal{P}$ be a base for $X$, and let $\mathcal{U} = \{\{x\}\colon\ x \in X\}$. Since $X$ has no non-trivial convergent sequence [4], $\mathcal{U}$ is an so-cover of $X$. Put $\mathcal{P}' = \{P \in \mathcal{P}\colon\ P \subset U \text{ for some } U \in \mathcal{U}\}$. Then $\mathcal{P}' = \{\{x\}\colon\ x \in \mathbb{N}\}$ is not a cs-network of $X$.

**Lemma 3.3.**

(1) *A space has a countable cs-network if and only if it is a sequence-covering image of a separable metric space* [27].

(2) *A space has a point-countable cs-network if and only if it is a sequence-covering s-image of a metric space* [14].

**Theorem 3.4.** *The following are equivalent for a space $X$:*

(1) *$X$ is a sequence-covering and s-image of a locally separable metric space.*

(2) *$X$ has a point-countable cs-network consisting of cosmic subspaces.*

(3) *$X$ has a point-countable cs-network, and an so-cover consisting of $\aleph_0$-subspaces.*

**Proof.** (1) ⇒ (2) Let $f : M \to X$ be a sequence-covering and s-map, here $M$ is a locally separable metric space. Suppose $\mathcal{B}$ is a $\sigma$-locally finite base for $M$ consisting of separable subspaces. Put $\mathcal{P} = \{f(B)\colon\ B \in \mathcal{B}\}$. Then $\mathcal{P}$ is a point-countable cs-network for $X$ consisting of cosmic subspaces.

(2) ⇒ (3) Let $\mathcal{P}$ be a point-countable cs-network of $X$ consisting of cosmic subspaces. For each $P \in \mathcal{P}$, let $D(P)$ be a countable and sequentially dense subset of $P$. For each $x \in X$, put

$$\mathcal{P}(x,1) = \{P \in \mathcal{P}\colon\ x \in P\}, \qquad D(x,1) = \bigcup \{D(P)\colon\ P \in \mathcal{P}(x,1)\},$$

and for each $n \geqslant 2$ inductively define that

$$\mathcal{P}(x,n) = \{P \in \mathcal{P}\colon\ P \cap D(x,n-1) \neq \emptyset\},$$

$$D(x,n) = \bigcup \{D(P)\colon\ P \in \mathcal{P}(x,n)\}.$$

Let $\mathcal{P}(x) = \bigcup\{\mathcal{P}(x,n)\colon\ n \in \mathbb{N}\}$, and $U(x) = \bigcup \mathcal{P}(x)$. To complete the proof of (3), it suffices to show that $U(x)$ is sequentially open in $X$ and $\mathcal{P}(x)$ is a cs-network for $U(x)$. If $\{y_n\}$ is a sequence in $X$ converging to a point $y \in U(x) \cap W$ with $W$ open in $X$, then $y \in P$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}(x,m)$, and there is a sequence $\{z_n\}$ in $D(P)$ with $z_n \to y$, thus $\{y\} \cup \{y_n, z_n\colon\ n \geqslant m\} \subset Q \subset W$ for some $m \in \mathbb{N}$ and some $Q \in \mathcal{P}$, so

$Q \in \mathcal{P}(x, m+1) \subset \mathcal{P}(x)$ and $\{y\} \cup \{y_n\colon n \geqslant m\} \subset Q \subset U(x) \cap W$. This implies that $U(x)$ is sequentially open and $\mathcal{P}(x)$ is a cs-network for $U(x)$.

(3) $\Rightarrow$ (1) By Lemma 3.2, $X$ has a point-countable cs-network $\mathcal{P}$ consisting of $\aleph_0$-subspaces. Let $\mathcal{P} = \{P_\alpha\colon \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, in view of Lemma 3.3, there are a separable metric space $M_\alpha$ and a sequence-covering map $f_\alpha : M_\alpha \to P_\alpha$. Put

$$M = \bigoplus_{\alpha \in \Lambda} M_\alpha, \quad Z = \bigoplus_{\alpha \in \Lambda} P_\alpha \quad \text{and} \quad f = \bigoplus_{\alpha \in \Lambda} f_\alpha : M \to Z.$$

Then $M$ is a locally separable metric space and $f$ is a sequence-covering map. Define $h : Z \to X$ a natural map, and let $g = h \circ f : M \to X$. Then $g$ is a sequence-covering and s-map. $\square$

(1) $\Leftrightarrow$ (2) in Theorem 3.4 is proved in [33]. A role about (3) is that a decomposition of the sequence-covering, quotient and s-image of a locally separable metric space can be given by it. It is also closely related to another question posed in [15]: Is every quotient and s-image of a locally separable metric space equivalent to a quotient and s-image of a metric space and its each first countable subspace is locally separable?

Recall some basic definitions. Let $f : X \to Y$ be a map. $f$ is *quotient* if whenever $f^{-1}(U)$ is open in $X$, then $U$ is open in $Y$. $f$ is *pseudo-open* if whenever $f^{-1}(y) \subset V$ with $V$ open in $X$, then $y \in \text{int}(f(V))$. It is showed that $f$ is a sequentially quotient if and only if whenever $f^{-1}(U)$ in sequentially open in $X$, then $U$ is sequentially open in $Y$ [2].

**Lemma 3.5** [2,5]. *Let $f : X \to Y$ be a map.*

(1) *If $X$ is sequential, then $f$ is quotient if and only if $Y$ is sequential and $f$ is sequentially quotient.*
(2) *If $X$ is Fréchet, then $f$ is pseudo-open if and only if $Y$ is Fréchet and $f$ is sequentially quotient.*

**Corollary 3.6.** *The following are equivalent for a space $X$:*

(1) *$X$ is a sequence-covering, quotient and s-image of a locally separable metric space.*
(2) *$X$ is a local $\aleph_0$-space and a sequence-covering, quotient and s-image of a metric space.*
(3) *$X$ is a sequential and local $\aleph_0$-space with a point-countable cs-network.*

Now, we further investigate the condition in which "so-cover" in Theorem 3.4 is point-countable.

**Theorem 3.7.** *The following are equivalent for a space $X$:*

(1) *$X$ has a star-countable cs*-network (cs-network).*
(2) *$X$ has a point-countable so-cover consisting of $\aleph_0$-subspaces.*
(3) *$X$ has a disjoint so-cover consisting of $\aleph_0$-subspaces.*

**Proof.** (1) $\Rightarrow$ (3) Let $\mathcal{P}$ be a star-countable cs*-network for $X$. By Lemma 3.10 in [3], $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha\colon \alpha \in \Lambda\}$, here each $\mathcal{P}_\alpha$ is countable and $(\bigcup \mathcal{P}_\alpha) \cap (\bigcup \mathcal{P}_\beta) \neq \emptyset$ if and only if

$\alpha \neq \beta$. For each $\alpha \in \Lambda$, let $X_\alpha = \bigcup \mathcal{P}_\alpha$, and $\mathcal{R}_\alpha = \{\bigcup P'\colon\ P'$ is a finite subfamily of $\mathcal{P}_\alpha\}$. Then $X_\alpha$ is sequentially open and $\mathcal{R}_\alpha$ is a countable cs-network for $X_\alpha$. Indeed, if a sequence $\{x_n\}$ in $X$ converges to a point $x \in X_\alpha \cap U$ with $U$ open in $X$, let $\mathcal{R} = \{P \in \mathcal{P}\colon\ x \in P \subset U\} = \{P_i\colon\ i \in \mathbb{N}\}$, then $\{x_n\colon\ n \geqslant m\} \subset \bigcup_{i \leqslant k} P_i$ for some $m, k \in \mathbb{N}$ because $\mathcal{P}$ is a cs*-network for $X$, thus $\bigcup_{i \leqslant k} P_i \in \mathcal{R}_\alpha$, and $\bigcup_{i \leqslant k} P_i \subset X_\alpha \cap U$. Hence, $\{X_\alpha\colon\ \alpha \in \Lambda\}$ is a disjoint so-cover consisting of $\aleph_0$-subspaces.

(3) $\Rightarrow$ (2) Obviously.

(2) $\Rightarrow$ (1) Let $\mathcal{P}$ be a point-countable so-cover for $X$ consisting of $\aleph_0$-subspaces. Put $\mathcal{P} = \{P_\alpha\colon\ \alpha \in \Lambda\}$. For each $\beta \in \Lambda$, a countable and sequentially dense subset of $P_\beta$ is denoted by $D_\beta$. Since $P_\alpha$ is sequentially open for each $\alpha \in \Lambda$, $D_\beta \cap P_\alpha \neq \emptyset$ if and only if $P_\beta \cap P_\alpha \neq \emptyset$. Thus $\{P_\alpha \in \mathcal{P}\colon\ P_\beta \cap P_\alpha \neq \emptyset\}$ is countable. It follows that $\mathcal{P}$ is star-countable. Suppose $\mathcal{P}_\alpha$ is a countable cs-network of $P_\alpha$ for each $\alpha \in \Lambda$, then it is easy to show that $\bigcup\{\mathcal{P}_\alpha\colon\ \alpha \in \Lambda\}$ is a star-countable cs-network of $X$. □

**Corollary 3.8.** *The following are equivalent for a sequential space $X$*:

(1) *$X$ has a star-countable cs*-network* (*cs-network*).

(2) *$X$ has a point-countable open cover consisting of $\aleph_0$-subspaces.*

(3) *$X$ is a topological sum of $\aleph_0$-subspaces.*

(4) *$X$ is a metalindelöf and a local $\aleph_0$-space.*

**Corollary 3.9** [11,34]. *The following are equivalent for a Fréchet space $X$*:

(1) *$X$ is a quotient and s-image of a locally separable metric space.*

(2) *$X$ is a locally separable space and a quotient and s-image of a metric space.*

(3) *$X$ has a locally countable cs*-network* (*k-network, cs-network*).

**Proof.** (1) $\Rightarrow$ (2) Observe that local separability is preserved by a pseudo-open and s-map. By Lemma 3.5, $X$ is locally separable.

(2) $\Rightarrow$ (3) $X$ is a local $\aleph_0$-space by Remark 2.7(4), and $X$ is a metalindelöf space by Proposition 8.6 in [9]. By (3) $\Leftrightarrow$ (4) in Corollary 3.8, $X$ has a locally countable cs-network.

(3) $\Rightarrow$ (1) $X$ is a local $\aleph_0$-space and a metalindelöf space by Proposition 8.6 in [9]. By Corollaries 3.8 and 3.6, $X$ is a quotient and s-image of a locally separable metric space. □

**Remark 3.10.**

(1) By a similar method in the proof of (1) $\Rightarrow$ (3) in Theorem 3.7, it can be proved that spaces with a locally countable cs*-network are equivalent to spaces with a locally countable k-network, and spaces with a locally countable cs-network.

(2) A space with a star-countable cs-network $\not\Rightarrow$ a space with a star-countable k-network; see Example $\beta\mathbb{N}$.

(3) A quotient and s-image of a locally separable metric space, which has a star-countable k-network $\not\Rightarrow$ locally separable; see Example 9.8 in [9] or Example 2.9.27 in [11].

(4) A sequential space with a point-countable cs-network consisting of cosmic subspace $\not\Rightarrow$ a space with a point-countable so-cover consisting of cosmic subspace; see Example 9.3 in [9] or Example 2.8.16 in [11].

(5) The condition “$\aleph_0$-subspaces” in Theorem 3.7 cannot be replaced by “cosmic subspaces” because a cosmic space $\not\Rightarrow$ a space with a point-countable cs-network; see Example 1.8.3 in [11] or the “butterfly space” in [20].

**Question 3.11.** Is a separable and sequence-covering, quotient and s-image of a metric space a local $\aleph_0$-space?

## 4. A sequence-covering and compact image of a metric space

**Definition 4.1** [12]. Let $f : X \to Y$ be a map. $f$ is a 1-*sequence-covering map* if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$ in $Y$ there is a sequence $\{x_n\}$ converging to $x$ in $X$ with each $x_n \in f^{-1}(y_n)$.

Every open map of a first countable space is 1-sequence-covering [12].

**Definition 4.2.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_x\colon x \in X\}$ be a cover of a space $X$ such that for each $x \in X$,

(1) $\mathcal{P}_x$ is a *network of* $x$ in $X$, i.e., $x \in \bigcap \mathcal{P}_x$ and for $x \in U$ with $U$ open in $X$, $P \subset U$ for some $P \in \mathcal{P}_x$.

(2) If $U, V \in \mathcal{P}_x$, $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is a *weak base* [1] for $X$ if whenever $G \subset X$ satisfying for each $x \in G$ there is $P \in \mathcal{P}_x$ with $P \subset G$, then $G$ is open in $X$. $\mathcal{P}$ is an *sn-network* [12,13] for $X$ if each element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ in $X$ for each $x \in X$, here $\mathcal{P}_x$ is an *sn-network of* $x$ in $X$.

For a space, weak base $\Rightarrow$ sn-network $\Rightarrow$ cs-network. An sn-network for a sequential space is a weak base [12,13]. The purpose introduced 1-sequence-covering maps is to obtain a characterization of a space with a point-countable weak base.

**Lemma 4.3** [12].

(1) *A space is a* 1*-sequence-covering and s-image of a metric space if and only if it has a point-countable sn-network.*

(2) *A space is a* 1*-sequence-covering, quotient and s-image of a metric space if and only if it has a point-countable weak base.*

By Lemmas 3.3 and 4.3, the sequential fan $S_\omega$ is a sequence-covering and s-image of a metric space, and cannot be a 1-sequence-covering and s-image of a metric space. As an answer for Question 1.2, in [16] we showed that for a space $X$, the following conditions are equivalent:

(1) $X$ is a sequence-covering, quotient and compact image of a metric space.

(2) $X$ is a 1-sequence-covering, quotient and compact image of a metric space.

(3) $X$ is a sequential space with a point-regular cs-network.

(4) $X$ has a point-regular weak base.

Here a family $\mathcal{P}$ of a space $X$ is a point-regular cover [4] if for each $x \in U$ with $U$ open in $X$, $\{P \in \mathcal{P}\colon\ x \in P \not\subset U\}$ is finite. In this section, a further result about sequence-covering and compact maps of metric spaces is proved as follows.

**Theorem 4.4.** *Let $f : X \to Y$ be a sequence-covering and compact map. If $X$ is a metric space, $f$ is a* 1*-sequence-covering map.*

**Proof.** Since $X$ is a metric space, there is a locally finite sequence $\{\mathcal{B}_n\}$ of open covers of $X$ satisfying [4,11]:

(a) each $\mathcal{B}_{n+1}$ is a star refinement of $\mathcal{B}_n$, i.e., for each $B \in \mathcal{B}_{n+1}$, there is $C \in \mathcal{B}_n$ such that $\mathrm{st}(B, \mathcal{B}_{n+1}) \subset C$.

(b) $\{\mathcal{B}_n\}$ is a development for $X$.

For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{f(B)\colon\ B \in \mathcal{B}_n\}$, then

(c) For each $z \in Y$, there is $P_z \in \mathcal{P}_n$ such that $P_z$ is a sequential neighborhood of $z$ in $Y$.

Since $f$ is compact, $\mathcal{P}_n$ is a point-finite cover of $Y$. Put $(\mathcal{P}_n)_z = \{P_n\colon\ i \leqslant k\}$. For each $i \leqslant k$, if $P_n$ is not a sequential neighborhood of $z$ in $Y$, there is a sequence $\{z_{in}\}_n$ converging to $z$ in $Y$ with any $z_{in} \notin P_n$. Define $z_m = z_{in}$ with $m = (n-1)k + i$ and $i \leqslant k$, then $z_m \to z$. There is a sequence $\{x_m\}$ converging to some point $x \in f^{-1}(z)$ with each $x_m \in f^{-1}(z_m)$ because $f$ is sequence-covering. Take $B \in (\mathcal{B}_n)_x$, then $x_m \in B$ when $m \geqslant m_0$ for some $m_0 \in \mathbb{N}$, and $P_i = f(B)$ for some $i \leqslant k$, thus $z_{in} \in P_i$ when $n \geqslant m_0$, a contradiction.

For each $y_0 \in Y$, put

$$U_n = \bigl\{x \in X\colon\ f(B) \text{ is not a sequential neighborhood of } y_0 \text{ in } Y \text{ for each } B \in (\mathcal{B}_n)_x\bigr\}.$$

Then

(d) If $x \in U_n$, $\bigcap(\mathcal{B}_{n+1})_x \subset U_{n+1}$.

If not, choose a point $p \in \bigcap(\mathcal{B}_{n+1}) \setminus U_{n+1}$, then $f(B)$ is a sequential neighborhood of $y_0$ in $Y$ for some $B \in (\mathcal{B}_{n+1})_p$ by the definition of $U_{n+1}$. Take some $B_1 \in (\mathcal{B}_{n+1})_x$, then $p \in B \cap B_1$, thus $B \cup B_1 \subset B_2$ for some $B_2 \in \mathcal{B}_n$ by (a), so $B_2 \in (\mathcal{B}_n)_x$ and $f(B_2)$ is a sequential neighborhood of $y_0$ in $Y$, hence $x \notin U_n$, a contradiction.

(e) $f^{-1}(y_0) \not\subset \bigcup\{U_n\colon\ n \in \mathbb{N}\}$.

If not, $f^{-1}(y_0) \subset \bigcup\{U_n\colon\ n \in \mathbb{N}\}$. By (d), for each $n \in \mathbb{N}$,

$$U_n \subset \bigcup\Bigl\{\bigcap(\mathcal{B}_{n+1})_x\colon\ x \in U_n\Bigr\} \subset U_{n+1}.$$

Since $f^{-1}(y_0)$ is compact and $\bigcap(\mathcal{B}_{n+1})_x$ is open in $X$, $f^{-1}(y_0) \subset U_m$ for some $m \in \mathbb{N}$. By (c), there is $B \in \mathcal{B}_m$ such that $f(B)$ is a sequential neighborhood of $y_0$ in $Y$, then $\emptyset \neq f^{-1}(y_0) \cap B \subset X \setminus U_m$, a contradiction.

Now, fix a point $x_0 \in f^{-1}(y_0) \setminus \bigcup\{U_n\colon\ n \in \mathbb{N}\}$, then

(f) If $y_i \to y_0$ in $Y$, there is $x_i \in f^{-1}(y_i)$ for each $i \in \mathbb{N}$ with $x_i \to x_0$ in $X$.

For each $n \in \mathbb{N}$, there is $B_n \in (\mathcal{B}_n)_{x_0}$ such that $f(B_n)$ is a sequential neighborhood of $y_0$ in $Y$ by $x_0 \notin U_n$, then $y_i \in f(B_n)$ when $i \geqslant i(n)$ for some $i(n) \in \mathbb{N}$, thus $B_n \cap f^{-1}(y_i) \neq \emptyset$.

We can assume that $1 < i(n) < i(n+1)$. For each $j \in \mathbb{N}$, choose

$$x_j \in \begin{cases} f^{-1}(y_j), & j < i(1), \\ f^{-1}(y_j) \cap B_n, & i(n) \leqslant j < i(n+1),\ n \in \mathbb{N}. \end{cases}$$

Then $x_j \in f^{-1}(y_j)$, and $x_j \to x_0$ in $X$ by (b).

In a word, $f$ is a 1-sequence-covering map. □

Finally, we discuss some relations among the sequence-covering and compact images of separable metric spaces. Let $\mathcal{P}$ be a cover of a space $X$. $\mathcal{P}$ has a *CFP-property* (i.e., *compact finite partition property*) [35] if whenever $K$ is compact in $X$, there are a finite collection $\{K_n\colon n \leqslant k\}$ of closed subsets of $K$ and $\{P_n\colon n \leqslant k\} \subset \mathcal{P}$ such that $K = \bigcup\{K_n\colon n \leqslant k\}$ and each $K_n \subset P_n$. The following lemma is due to [35].

**Lemma 4.5.** *A space $X$ is a compact-covering and compact image of a* (*separable*) *metric space if and only if there is a sequence $\mathcal{U}_n$ of* (*countable and*) *point-finite covers of $X$ such that*

(1) *each $\mathcal{U}_n$ is CFP,*

(2) $\{\mathrm{st}(x, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ *is a network of $x$ in $X$ for each $x \in X$.*

**Theorem 4.6.** *The following are equivalent for a space $X$*:

(1) *$X$ is a sequentially quotient and compact image of a separable metric space.*

(2) *$X$ is a compact-covering and compact image of a separable metric space.*

(3) *$X$ has a countable sn-network.*

**Proof.** (2) ⇒ (1) Obviously.

(1) ⇒ (3) Let $f : M \to X$ be a sequentially quotient and compact map, here $M$ is a separable metric space. There is a countable and locally finite sequence $\{\mathcal{B}_n\}$ of open covers of $M$ such that [4,11]

(a) each $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$,

(b) $\{\mathrm{st}(K, \mathcal{B}_n)\colon n \in \mathbb{N}\}$ is a local base of $K$ in $X$ for each compact $K$ in $X$.

For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{f(B)\colon B \in \mathcal{B}_n\}$. Then $\mathcal{P}_n$ is a countable and point-finite cover of $X$. Let

$$\mathcal{H} = \big\{\mathrm{st}(x, \mathcal{P}_n)\colon x \in X,\ n \in \mathbb{N}\big\}.$$

Then $\mathcal{H}$ is countable. We shall show that $\mathcal{H}$ is an sn-network for $X$. For each $x \in U$ with $U$ open in $X$, $\mathrm{st}(f^{-1}(x), \mathcal{B}_n) \subset f^{-1}(U)$ for some $n \in \mathbb{N}$ by (b), then $\mathrm{st}(x, \mathcal{P}_n) \subset U$. If $\mathrm{st}(x, \mathcal{P}_m)$ is not a sequential neighborhood of $x$ in $X$ for some $m \in \mathbb{N}$, there is a sequence $\{x_n\}$ converging to the point $x$ in $X$ with any $x_n \notin \mathrm{st}(x, \mathcal{P}_m)$, then there are a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $\alpha_i \in f^{-1}(x_{n_i})$ for each $i \in \mathbb{N}$ such that $\alpha_i \to \alpha \in f^{-1}(x)$ in $M$. Take $B \in (\mathcal{B}_m)_\alpha$, thus $\alpha_i \in B$ when $i \geqslant j$ for some $j \in \mathbb{N}$, so $x_n \in f(B) \subset \mathrm{st}(x, \mathcal{P}_m)$, a contradiction. Consequently, $\mathcal{H}$ is a countable sn-network for $X$.

(3) ⇒ (2) Let $\mathcal{P}$ be a countable sn-network for $X$. We can assume that each element of $\mathcal{P}$ is closed in $X$. Denote that

$$\mathcal{P} = \{P_n\colon n \in \mathbb{N}\} = \bigcup\{\mathcal{P}_x\colon x \in X\},$$

here each $\mathcal{P}_x$ is an sn-network of $x$ in $X$. For each $n \in \mathbb{N}$, put

$$Q_n = \{x \in X\colon\ P_n \notin \mathcal{P}_x\},$$
$$\mathcal{U}_n = \{P_n, Q_n\}.$$

Then $\mathcal{U}_n$ is a cover of $X$, and for each $x \in X$,

$$\mathrm{st}(x, \mathcal{U}_n) = \begin{cases} P_n, & P_n \in \wp_x, \\ X, & P_n \notin \wp_x,\ x \in P_n, \\ Q_n, & P_n \notin \wp_x,\ x \notin P_n. \end{cases}$$

Thus $\{\mathrm{st}(x, \mathcal{U}_n)\colon\ n \in \mathbb{N}\}$ is a network of $x$ in $X$. Suppose $C$ is compact in $X$, put

$$C_1 = P_n \cap C, \qquad C_2 = \overline{C \setminus P_n}.$$

Then $C = C_1 \cup C_2$. If $x \in C_2$, there is a sequence $\{x_i\}$ in $C \setminus P_n$ with $x_i \to x$ in $C$ because $C$ is metrizable in view of [7, Theorem 2.13], then $P_n \notin \mathcal{P}_x$, and $x \in Q_n$. Thus $C_2 \subset Q_n$ and $C_1 \subset P_n$. Hence $X$ is a compact-covering and compact image of a separable metric space by Lemma 4.5. □

A space has a countable weak base if and only if it is a sequential space with a countable sn-network [12].

**Corollary 4.7.** *The following are equivalent for a space* $X$:
1. (1) $X$ *is a quotient and compact image of a separable metric space.*
2. (2) $X$ *is a compact-covering, quotient and compact image of a separable metric space.*
3. (3) $X$ *has a countable weak base.*

We recall that a space $X$ is a *k-space* if whenever $A \subset X$ such that $A \cap K$ is closed for each compact $K$ in $X$, then $A$ is closed in $X$. Every sequential space is a k-space.

**Corollary 4.8.** *The following are equivalent for a k-space with a star-countable k-network*:
1. (1) $X$ *is a quotient and compact image of a locally separable metric space.*
2. (2) $X$ *is a compact-covering, quotient and compact image of a locally separable metric space.*
3. (3) $X$ *is a quotient and compact image of a metric space.*
4. (4) $X$ *is a compact-covering, quotient and compact image of a metric space.*
5. (5) $X$ *contains no closed copy of* $S_\omega$.

**Proof.** It only need to show that $(5) \Rightarrow (1)$. This is as in the proof of Theorems 4 and 5 in [19]. □

**Remark 4.9.**
1. (1) A space with a countable weak base $\not\Rightarrow$ a sequence-covering and compact image of a separable metric space; see Example 2.14(3) in [31].

(2) A perfect map of a compact metric space $\not\Rightarrow$ a sequence-covering map; see [25].
(3) A compact-covering, quotient and compact image of a locally compact metric space $\not\Rightarrow$ a space with a point-countable cs-network; see Example 9.8 in [9] or Example 2.9.27 in [11].

**Question 4.10.** Is a Fréchet space with a countable cs-network a closed and sequence-covering image of a separable metric space?

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